Chapter 16
Glossaries

Box 16.1 Glossary of New Concepts in Chapter 1

$PFE_{xx'}$ — *Prima facie effect* of treatment x compared to treatment x'. It is defined by

$$PFE_{xx'} \equiv E(Y \mid X{=}x) - E(Y \mid X{=}x').$$

$PFE_{xx';Z=z}$ — $(Z{=}z)$-*Conditional prima facie effect* of treatment x compared to treatment x'. It is defined by

$$PFE_{xx';Z=z} \equiv E(Y \mid X{=}x, Z{=}z) - E(Y \mid X{=}x', Z{=}z).$$

$E(PFE_{xx';Z})$ — *Average of* $(Z{=}z)$-*conditional prima facie effects* of treatment x compared to treatment x' with respect to Z. It is defined by

$$E(PFE_{xx';Z}) \equiv \sum_z PFE_{xx';Z=z} \cdot P(Z{=}z).$$

Box 16.2 Glossary of New Concepts in Chapter 2

Single-unit trial	The kind of random experiment that causal and ordinary stochastic dependencies refer to.
Potential Confounder	A variable that can never be affected by the putative cause. It is prior or simultaneous to the putative cause. It might be correlated with the putative cause and the outcome variable.
Covariate	A specific potential confounder that is used to consider conditional dependencies given one of its values.
Fallible covariate	A covariate that is measured with measurement error.
Latent covariate	A covariate that is not directly observed. Instead it is defined by a set of manifest variables and a measurement model describing their relationship to the latent covariate.
Potential mediator	A variable that might mediate (transmit) the effect of the putative cause on the outcome variable. The putative cause is always prior to a potential mediator and the potential mediator is always prior to the outcome variable.
Mediator	A potential mediator that actually transmits the effect of the putative cause on the outcome variable.

Box 16.3 Glossary of New Concepts in Chapter 3

Ω	Set of *possible outcomes* of the random experiment considered. It often is a Cartesian product of several other sets.
$\mathfrak{A}$	σ-algebra of *possible events* considered in the random experiment. In simple cases, it is the power set of Ω.
$(\Omega, \mathfrak{A})$	*Measurable space*. It consists of a set Ω and a σ-algebra $\mathfrak{A}$ on Ω.
P	*Probability measure*. It assigns to each event A its probability $P(A)$.
P_B	*Conditional-probability measure*. It assigns to each $A \in \mathfrak{A}$ its B-conditional probability $P_B(A) = P(A \cap B) / P(B)$.
$(\Omega, \mathfrak{A}, P)$	*Probability space*. It represents the random experiment considered.
X	*Random variable* $X\colon (\Omega, \mathfrak{A}) \to (\Omega', \mathfrak{A}')$ on $(\Omega, \mathfrak{A}, P)$. It is defined to be a *measurable mapping*, i. e., a mapping such that $X^{-1}(A') \in \mathfrak{A}$, for each $A' \in \mathfrak{A}'$. This implies that X has a distribution P^X.
P^X	*Distribution* of the random variable $X\colon (\Omega, \mathfrak{A}) \to (\Omega', \mathfrak{A}')$. It assigns the probability $P^X(A') = P[X^{-1}(A')]$ to the event that X takes a value in a set $A' \in \mathfrak{A}'$.
$X^{-1}(\mathfrak{A}')$	σ-*algebra generated by* X and the σ-algebra $\mathfrak{A}'$ on Ω', i. e., the set of all inverse images $X^{-1}(A') \equiv \{\omega \in \Omega\colon X(\omega) \in A'\}$, $A' \in \mathfrak{A}'$.
$\sigma(X)$	Same as $X^{-1}(\mathfrak{A}')$. This notation can be used if it is clear to which σ-algebra $\mathfrak{A}'$ we refer to.
$\sigma(\mathfrak{E})$	σ-*algebra generated* by a set $\mathfrak{E}$ of subsets of Ω. It is the intersection of all σ-algebras on Ω containing $\mathfrak{E}$ as a subset.
$E(X)$	*Expectation* of a numerical random variable.
$Var(X)$	*Variance* of a numerical random variable.
$Std(X)$	*Standard deviation* of a numerical random variable. It also describes the variability of X. It is the positive square root of $Var(X)$.
$Cov(X, Y)$	*Covariance* of two numerical random variables. It describes the strength of the dependence of Y on X described by $Q(Y \mid X)$.
$Corr(X, Y)$	*Correlation* of two numerical random variables. It also describes the strength of the dependence described by $Q(Y \mid X)$. In contrast to $Cov(X, Y)$, the correlation only takes values between -1 and 1.
$Q(Y \mid X)$	*Linear least-squares regression* of Y on X. It is the linear function $\alpha_0 + \alpha_1 X$ of the regressor X that minimizes the least-squares function $LS(a_0, a_1) = E\big([Y - (a_0 + a_1 X)]^2\big)$.

Box 16.4 Glossary of New Concepts in Chapter 3 continued

$E(Y \mid X{=}x)$	*Conditional expectation* of a numerical random variable Y given the value x of another random variable X.
$E(Y \mid \mathfrak{C})$	The *conditional expectation* of Y given the σ-algebra $\mathfrak{C}$.
$E(Y \mid X)$	A special case of $E(Y \mid \mathfrak{C})$ with $\mathfrak{C} = X^{-1}(\mathfrak{A}')$. It is called the *regression* of Y on X. It is the random variable on $(\Omega, \mathfrak{A}, P)$ whose values are the conditional expectations $E(Y \mid X{=}x)$.
$E_B(Y \mid \mathfrak{C})$	The *conditional expectation* of Y given the σ-algebra $\mathfrak{C}$ w. r. t. the conditional-probability measure P_B.
$E_B(Y \mid Z)$	The B-*conditional regression* of Y on Z. It is the regression of Y on Z w. r. t. the B-conditional-probability measure P_B.
$E_{X=x}(Y \mid Z)$	If $B = \{X{=}x\} \equiv \{\omega \in \Omega\colon X(\omega) = x\}$, then $E_B(Y \mid Z)$ is also denoted by $E_{X=x}(Y \mid Z)$ and called the $(X{=}x)$-*conditional regression* of Y on Z.
$P^{Y \mid X=x}$	*Conditional distribution* of Y given $X{=}x$. It can be used to describe how the distribution of Y depends on the values x of X.
$X_1 \perp\!\!\!\perp X_2$	Independence of the random variables X_1 and X_2.
$X_1 \perp\!\!\!\perp X_2 \mid Z$	Z-conditional independence of the random variables X_1 and X_2.
$\mathfrak{C}_1 \perp\!\!\!\perp \mathfrak{C}_2$	Independence of the two sets of events $\mathfrak{C}_1$ and $\mathfrak{C}_2$.
$\mathfrak{C}_1 \perp\!\!\!\perp \mathfrak{C}_2 \mid \mathfrak{D}$	$\mathfrak{D}$-conditional stochastic independence of the two sets of events $\mathfrak{C}_1$ and $\mathfrak{C}_2$.

Box 16.5 Glossary of New Concepts in Chapter 4

$(\mathfrak{C}_t)_{t\in T}$	*Filtration.* A family of σ-algebras on the same set Ω such that $\mathfrak{C}_s \subset \mathfrak{C}_t$, if $s \le t$, where $s, t \in T$. It represents the process that allows us to define pre- and equi-orderedness of events, random variables and σ-algebras.
Regular filtration	A filtration is *regular* w. r. t. X if there is a $t_X \in T$ such that $X^{-1}(\mathfrak{A}'_X) \subset \mathfrak{C}_{t_X}$ and $X^{-1}(\mathfrak{A}'_X) \not\subset \mathfrak{C}_t$, if $t < t_X$.
$\mathfrak{C}_{t_X}$	*X-concurrent σ-algebra w. r. t.* $(\mathfrak{C}_t)_{t\in T}$. The σ-algebra $\mathfrak{C}_t$ of the filtration $(\mathfrak{C}_t)_{t\in T}$ that is equi-ordered to X.
$\mathfrak{D}_X$	*Potential confounder σ-algebra.* A σ-algebra that is pre- or equi-ordered to the putative cause X. Together with X it generates $\mathfrak{C}_{t_X}$ and contains as subsets all σ-algebras of the filtration $(\mathfrak{C}_t)_{t\in T}$ that are pre-ordered to X.
Potential confounder w. r. t. $\mathfrak{D}_X$	A random variable that is measurable w. r. t. the confounder σ-algebra $\mathfrak{D}_X$. If W_1 and W_2 are potential confounders w. r. t. $\mathfrak{D}_X$, then each measurable function of (W_1, W_2) is a potential confounder w. r. t. X as well.
Covariate w. r. t. X	A potential confounder on which we condition in a regression or in a conditional distribution.
$\langle(\Omega, \mathfrak{A}, P), (\mathfrak{C}_t)_{t\in T}, X, Y, \mathfrak{D}_X\rangle$	*Causality space.* It summarizes the mathematical structures and assumptions under which we can meaningfully define causal effects and raise the question if the dependency of Y on X describes a causal dependency. It consists of a probability space $(\Omega, \mathfrak{A}, P)$, a filtration $(\mathfrak{C}_t)_{t\in T}$ that is regular w. r. t. X, the focused putative cause X, the outcome variable Y, and the potential-confounder σ-algebra $\mathfrak{D}_X$. It is assumed that X is pre-ordered to Y.
Potential mediator of X	A random variable such that X is pre-ordered to M that itself is pre-ordered to the outcome variable Y.
Pre-orderedness	A random variable X is *pre-ordered* to another random variable, say Y, w. r. t. $(\mathfrak{C}_t)_{t\in T}$, if there is an $s \in T$ such that the σ-algebra generated by X is a subset of $\mathfrak{C}_s$, the σ-algebra generated by Y is not a subset of $\mathfrak{C}_s$, and there is a $t \in T$, $s < t$, such that the σ-algebra generated by Y is a subset of $\mathfrak{C}_t$.
Equi-orderedness	A random variable X is *equi-ordered* to a random variable Y, w. r. t. $(\mathfrak{C}_t)_{t\in T}$, if there is a $t \in T$ such that the σ-algebras generated by X and by Y, respectively, are subsets of $\mathfrak{C}_t$, but there is no $s \in T$, $s < t$, such that the σ-algebra generated by X or the σ-algebra generated by Y is a subset of $\mathfrak{C}_s$.

Box 16.6 Glossary of New Concepts in Chapter 5

τ_x	The *true outcome variable of* $X{=}x$ *with respect to total effects*. Presuming that $E_{X=x}(Y\,|\,\mathfrak{D}_X)$ is P-unique, this random variable is defined by $\tau_x \equiv E_{X=x}(Y\,|\,\mathfrak{D}_X)$, i. e., by the conditional expectation of Y given $\mathfrak{D}_X$ w. r. t. the conditional-probability measure $P_{X=x}$. If $E_{X=x}(Y\,|\,\mathfrak{D}_X)$ is not P-unique, we may still use the term τ_x. However, in this case we will not call it the true outcome variable. Because $\mathfrak{D}_X$ denotes a potential-confounder σ-algebra, τ_x is constructed such that it is unconfounded. Note that τ_x is not necessarily unique. Instead, τ_x may only be a one out of many true outcome variables pertaining to x. Therefore, we also call it a *version* of the true outcome variable with respect to total effects.
$\delta_{xx'}$	The *true total effect variable*. Presuming that $\tau_x \equiv E_{X=x}(Y\,|\,\mathfrak{D}_X)$ and $\tau_{x'} \equiv E_{X=x'}(Y\,|\,\mathfrak{D}_X)$ are P-unique, it is defined by $\delta_{xx'} \equiv \tau_x - \tau_{x'}$. Its values are the true *total* effects pertaining to $X{=}x$ and $X{=}x'$ given the most fine-grained (or atomic) strata of potential confounders.
$\tau_{x;\,M}$	The *true outcome variable of* $X{=}x$ *with respect to direct effects*. Presuming that $E_{X=x}(Y\,|\,\mathfrak{D}_X, M)$ is P-unique, it is defined to be the conditional expectation $E_{X=x}(Y\,|\,\mathfrak{D}_X, M)$ of Y given $\mathfrak{D}_X$ and the mediator M w. r. t. the conditional-probability measure $P_{X=x}$. If X is a treatment variable, then $\tau_{x;\,M}$ is the conditional expectation of Y given $\mathfrak{D}_X$ and M in treatment x.
$\delta_{xx';\,M}$	The *true direct effect variable*. Presuming that $E_{X=x}(Y\,|\,\mathfrak{D}_X, M)$ and $E_{X=x'}(Y\,|\,\mathfrak{D}_X, M)$ are P-unique, it is defined to be the difference $\delta_{xx';\,M} \equiv \tau_{x;\,M} - \tau_{x';\,M}$. Its values are the true *direct* effects pertaining to $X{=}x$ and $X{=}x'$. These direct effects are *not* transmitted through the potential mediator M.
$\delta_{xx'} - \delta_{xx';\,M}$	The *true indirect effect variable*. Its values are the true *indirect* effects pertaining to $X{=}x$ and $X{=}x'$, provided that $E_{X=x}(Y\,|\,\mathfrak{D}_X, M)$ and $E_{X=x'}(Y\,|\,\mathfrak{D}_X, M)$ are P-unique. These indirect effects might be transmitted through the potential mediator M.

Box 16.7 Glossary of New Concepts in Chapter 6

In this box, we often refer to a random variable V. Examples for such a random variable V are a covariate Z, the putative cause X, the vector (X, Z), a mediator M, or the observational-unit variable U. The true outcome variables τ_x and $\tau_{x;M}$, and the true effect variables $\delta_{xx'}$ and $\delta_{xx';M}$ have been defined in chapter 5. Note these effects and effect functions can only be defined under appropriate uniqueness assumptions.

Difference Parametrization

Total Effects

$E(\delta_{xx'})$	*Average total effect of x vs. x′.*
$E(\delta_{xx'} \mid V{=}v)$	*$(V{=}v)$-conditional total effect of x vs. x′.*
$E(\delta_{xx'} \mid V)$	*V-conditional total effect function of x vs. x′.*

Direct Effects

$E(\delta_{xx';M})$	*Average direct effect of x vs. x′.*
$E(\delta_{xx';M} \mid V{=}v)$	*$(V{=}v)$-conditional direct effect of x vs. x′.*
$E(\delta_{xx';M} \mid V)$	*V-conditional direct effect function of x vs. x′.*

Indirect Effects

$E(\delta_{xx'} - \delta_{xx';M})$	*Average indirect effect of x vs. x′.*
$E(\delta_{xx'} - \delta_{xx';M} \mid V{=}v)$	*$(V{=}v)$-conditional indirect effect of x vs. x′.*
$E(\delta_{xx'} - \delta_{xx';M} \mid V)$	*V-conditional indirect effect function of x vs. x′.*

Effect Parametrization

Total Effects

$\delta_x \equiv \tau_x - \dfrac{1}{J+1}\sum_{x'=0}^{J} \tau_{x'}$	*True total effect variable of x.*
$E(\delta_x)$	*Average total effect* of x.
$E(\delta_x \mid V{=}v)$	*$(V{=}v)$-conditional total effect of x.*
$E(\delta_x \mid V)$	*V-conditional total effect function* of x.

Direct Effects

$\delta_{x;M} \equiv \tau_{x;M} - \dfrac{1}{J+1}\sum_{x'=0}^{J} \tau_{x';M}$	*True direct effect variable of x.*
$E(\delta_{x;M})$	*Average direct effect* of x.
$E(\delta_{x;M} \mid V{=}v)$	*$(V{=}v)$-conditional direct effect of x.*
$E(\delta_{x;M} \mid V)$	*V-conditional direct effect function of x.*

Indirect Effects

$\delta_x - \delta_{x;M}$	*True indirect effect variable of x.*
$E(\delta_x - \delta_{x;M})$	*Average indirect effect* of x.
$E(\delta_x - \delta_{x;M} \mid V{=}v)$	*$(V{=}v)$-conditional indirect effect of x.*
$E(\delta_x - \delta_{x;M} \mid V)$	*V-conditional indirect effect function of x.*

Box 16.8 Glossary of New Concepts in Chapter 7

All definitions presented in this box presume appropriate uniqueness assumptions (for details, see the definitions in the text).

Unbiasedness With Respect to Total Effects

Unbiasedness of $E(Y \mid X{=}x)$	$E(Y \mid X{=}x) = E(\tau_x)$.
Unbiasedness of $PFE_{xx'}$	$PFE_{xx'} = E(\delta_{xx'})$.
$(Z{=}z)$-conditional unbiasedness of $E(Y \mid X{=}x, Z{=}z)$	$E(Y \mid X{=}x, Z{=}z) = E(\tau_x \mid Z{=}z)$.
$(Z{=}z)$*-conditional unbiasedness of* $PFE_{xx';Z=z}$	$PFE_{xx';Z=z} = E(\delta_{xx'} \mid Z{=}z)$.
Z*-conditional unbiasedness of* $E_{X=x}(Y \mid Z)$	$E_{X=x}(Y \mid Z) \underset{a.s.}{=} E(\tau_x \mid Z)$.
Z*-conditional unbiasedness of* $PFE_{xx';Z}$	$PFE_{xx';Z} \underset{a.s.}{=} E(\delta_{xx'} \mid Z)$.

Unbiasedness With Respect to Direct Effects Relative to M

Unbiasedness of $E(Y \mid X{=}x, M{=}m)$	$E(Y \mid X{=}x, M{=}m) = E(\tau_{x;M} \mid M{=}m)$.
Unbiasedness of $PFE_{xx';M=m}$	$PFE_{xx';M=m} = E(\delta_{xx';M} \mid M{=}m)$.
Unbiasedness of $E_{X=x}(Y \mid M)$	$E_{X=x}(Y \mid M) \underset{a.s.}{=} E(\tau_{x;M} \mid M)$.
Unbiasedness of $PFE_{xx';M}$	$PFE_{xx';M} \underset{a.s.}{=} E(\delta_{xx';M} \mid M)$.
$(Z{=}z)$*-conditional unbiasedness of* $E(Y \mid X{=}x, Z{=}z, M{=}m)$	$E(Y \mid X{=}x, Z{=}z, M{=}m) = E(\tau_{x;M} \mid Z{=}z, M{=}m)$.
$(Z{=}z)$*-conditional unbiasedness of* $PFE_{xx';Z=z;M=m}$	$PFE_{xx';Z=z;M=m} = E(\delta_{xx';M} \mid Z{=}z, M{=}m)$.
Z*-conditional unbiasedness of* $PFE_{xx';Z;M}$	$PFE_{xx';Z;M} \underset{a.s.}{=} E(\delta_{xx';M} \mid Z, M)$.

Box 16.9 Glossary of New Concepts in Chapter 8

Unbiasedness With Respect to Total Effects

$E_{total}(Y \mid X)$	Purged regression of Y on X with respect to average total effects.
Unbiasedness of $E(Y \mid X)$	$E(Y \mid X) \underset{a.s.}{=} E_{total}(Y \mid X)$.
$E_{total}(Y \mid X; Z)$	Purged regression of Y on X and Z with respect to average total effects.
Z*-conditional unbiasedness of* $E(Y \mid X, Z)$	$E(Y \mid X, Z) \underset{a.s.}{=} E_{total}(Y \mid X; Z)$.

Unbiasedness With Respect to Direct Effects

$E_{direct}(Y \mid X.M)$	Purged regression of Y on X and M with respect to direct effects relative to the potential mediator M.
M*-conditional unbiasedness of* $E(Y \mid X, M)$	$E(Y \mid X, M) \underset{a.s.}{=} E_{direct}(Y \mid X.M)$
$E_{direct}(Y \mid X; Z.M)$	Purged regression of Y on X, Z, and M with respect to (Z, M)-conditional M-direct effects.
(Z, M)*-conditional unbiasedness of* $E(Y \mid X, Z, M)$	$E(Y \mid X, Z, M) \underset{a.s.}{=} E_{direct}(Y \mid X; Z.M)$.

Box 16.10 Glossary of New Concepts in Chapter 9

The two kinds of causality conditions summarized in this box presume that there is a causality space $\langle (\Omega, \mathfrak{A}, P), (\mathfrak{C}_t)_{t \in T}, X, Y, \mathfrak{D}_X \rangle$, a covariate Z, and a potential mediator M. Both kinds of causality conditions imply the unbiasedness.

Independent Cause Conditions

$X \perp\!\!\!\perp \mathfrak{D}_X$ — *Independence of X and the potential-confounder σ-algebra $\mathfrak{D}_X$.* This causality condition can be created by random assignment of units to treatment conditions.

$X \perp\!\!\!\perp \mathfrak{D}_X \mid Z$ — *Conditional independence of X and the potential-confounder σ-algebra $\mathfrak{D}_X$ given the covariate Z.* This causality condition can be created by conditional random assignment of units to treatment conditions based on the values z of Z. Units with different values z can be assigned to treatment conditions with different $(Z{=}z)$-conditional probabilities. Furthermore, the covariate $Z = (Z_1, \dots, Z_Q)$ can be selected such that $X \perp\!\!\!\perp \mathfrak{D}_X \mid Z$ might hold.

$X \perp\!\!\!\perp \mathfrak{D}_X \mid M$ — *Conditional independence of X and the potential-confounder σ-algebra $\mathfrak{D}_X$ given potential mediator M.* This causality condition can *not* be guaranteed by any design technique. It also can *not* be used for covariate selection.

$X \perp\!\!\!\perp \mathfrak{D}_X \mid Z, M$ — *Conditional independence of X and the potential-confounder σ-algebra $\mathfrak{D}_X$ given the covariate Z and the potential mediator M.* This causality condition can *not* be guaranteed by any design technique. However, the covariate $Z = (Z_1, \dots, Z_Q)$ can be selected such that $X \perp\!\!\!\perp \mathfrak{D}_X \mid Z, M$ might hold.

Complete Cause Conditions

$Y \vdash \mathfrak{D}_X \mid X$ — *Completeness of $E(Y \mid X)$ w.r.t. $\mathfrak{D}_X$.* This condition is defined by $E(Y \mid X, \mathfrak{D}_X) \underset{a.s.}{=} E(Y \mid X)$.

$Y \vdash \mathfrak{D}_X \mid X, Z$ — *Z-conditional completeness of $E(Y \mid X, Z)$ w.r.t. $\mathfrak{D}_X$.* It is defined by $E(Y \mid X, \mathfrak{D}_X) \underset{a.s.}{=} E(Y \mid X, Z)$. The covariate $Z = (Z_1, \dots, Z_Q)$ can be selected such that $Y \vdash \mathfrak{D}_X \mid X, Z$ might hold.

$Y \vdash \mathfrak{D}_X \mid X, M$ — *M-conditional completeness $E(Y \mid X, M)$ w.r.t. $\mathfrak{D}_X$.* It is defined by $E(Y \mid X, \mathfrak{D}_X, M) \underset{a.s.}{=} E(Y \mid X, M)$. This condition can *not* be utilized for any design technique.

$Y \vdash \mathfrak{D}_X \mid X, Z, M$ — *(Z, M)-conditional completeness $E(Y \mid X, Z, M)$ w.r.t. $\mathfrak{D}_X$.* It is defined by $E(Y \mid X, \mathfrak{D}_X, M) \underset{a.s.}{=} E(Y \mid X, Z, M)$. The covariate $Z = (Z_1, \dots, Z_Q)$ can be selected such that $Y \vdash \mathfrak{D}_X \mid X, Z, M$ might hold.