Analysis of Causal Effects

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Contents

- The problem: Simpson's paradox and similar phenomena
- Generalized ANCOVA
- EffectLite
- Theory of causal effects
- Identification of average causal effects

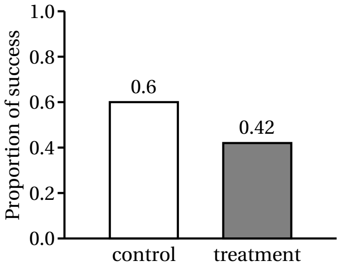
The Problem

Simpson's Paradox

Total sample

	Treatment		
Success	no (X=0)	yes (X=1)	
no (Y=0)	240	232	472
yes (Y=1)	360	168	528
	600	400	1000

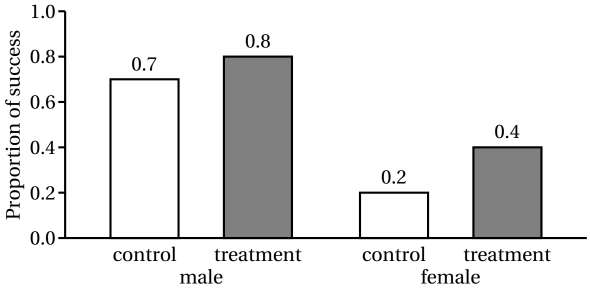
360/600 = .60 168/400 = .42



Simpson's Paradox: Males vs. Females

	Males (Z=0)		Females (Z=1)	
Success	Control (X=0)	Treatment (X=1)	Control (X=0)	Treatment (X=1)
No (Y=0)	144	4	96	228
Yes (Y=1)	336	16	24	152
	480	20	120	380

336/480 = .70   16/20 = .80   24/120 = .20   152/380 = .40



Nonorthogonal ANOVA

Table 1: Expected values in three treatment conditions

treatment	expected value of Y in the treatment conditions $E(Y \mid X{=}x)$	treatment probabilities $P(X{=}x)$
X=0 (control)	111.25	1/3
X=1 (treatment 1)	100.00	1/3
X=2 (treatment 2)	114.25	1/3
$E(Y)$	108.50	

Nonorthogonal ANOVA

Table 2: Expected values $E(Y \mid X{=}x, Z{=}z)$ in treatment and neediness conditions

	neediness						
treat-ment	low (Z=0)		medium (Z=1)		high (Z=2)		
X=0	120	(20/120)	110	(17/120)	60	(3/120)	(40/120)
X=1	100	(7/120)	100	(26/120)	100	(7/120)	(40/120)
X=2	80	(3/120)	90	(17/120)	140	(20/120)	(40/120)
		(30/120)		(60/120)		(30/120)	

Note. Probabilities $P(X{=}x, Z{=}z)$, $P(Z{=}z)$, and $P(X{=}x)$ in parentheses.

Traditional ANCOVA Model

If Z is a univariate covariate, the treatment variable X takes on values $0, 1, \dots, J$, and the random variables $I_{X=x}$ indicate with their values 1 and 0 whether or not $X{=}x$, *traditional analysis of covariance* assumes

$$E(Y \mid X, Z) = \gamma_{00} + \gamma_{01} \cdot Z + \sum_{x=1}^{J} \gamma_{x0} \cdot I_{X=x}. \tag{1}$$

For $X{=}0$, this equation yields:

$$E_{X=0}(Y \mid Z) = \gamma_{00} + \gamma_{01} \cdot Z, \tag{2}$$

and for $X{=}x$, Equation (1) yields:

$$E_{X=x}(Y \mid Z) = \gamma_{00} + \gamma_{01} \cdot Z + \gamma_{x0}\,. \tag{3}$$

Generalized ANCOVA Model

The fundamental equation for generalized analysis of covariance is:

$$E(Y \mid X, Z) = g_0(Z) + \sum_{x=1}^{J} g_x(Z) \cdot I_{X=x}, \tag{4}$$

where the *intercept function* $g_0(Z)$ and the *effect functions* $g_x(Z)$ are unknown functions of the (possibly multivariate, numerical or non-numerical) covariate Z.

Remember, *traditional analysis of covariance* assumes

$$E(Y \mid X, Z) = \gamma_{00} + \gamma_{01} \cdot Z + \sum_{x=1}^{J} \gamma_{x0} \cdot I_{X=x}. \tag{5}$$

Generalized ANCOVA Model

The fundamental equation for generalized analysis of covariance is:

$$E(Y \mid X, Z) = g_0(Z) + \sum_{x=1}^{J} g_x(Z) \cdot I_{X=x}, \tag{6}$$

where the *intercept function* $g_0(Z)$ and the *effect functions* $g_x(Z)$ are unknown functions of the (possibly multivariate, numerical or non-numerical) covariate Z.

Note that this equation is *always* true as long as no restrictive assumptions about the intercept and/or effect functions are introduced.

Conditioning on the covariate, Equation (6) yields

$$E_{Z=z}(Y \mid X) = g_0(z) + \sum_{x=1}^{J} g_x(z) \cdot I_{X=x}. \tag{7}$$

This equation shows that the effects of the treatments may be different for different values of the covariate.

Generalized ANCOVA Model

Conditioning on the treatment, Equation (6) yields, for $X{=}0$

$$E_{X=0}(Y \mid Z) \;=\; g_0(Z), \tag{8}$$

and for $X{=}x$:

$$E_{X=x}(Y \mid Z) \;=\; g_0(Z) + g_x(Z). \tag{9}$$

Hence,

$$E_{X=x}(Y \mid Z) - E_{X=0}(Y \mid Z) \;=\; g_x(Z) \tag{10}$$

is the treatment effect function.

EffectLite

Scope of *EffectLite*

EffectLite ...

- does not assume homogeneity of variances (in the univariate case with a single outcome variable) or of covariance matrices (in the multivariate case with two or more outcome variables) of the outcome variables between treatment groups.
- allows analyzing mean differences between groups with respect to several manifest outcome variables, one or more *latent outcome variables*, and a mixture of the two kinds of outcome variables.
- allows analyzing conditional and average effects with respect to several manifest covariates or with respect to one or more *latent covariates*, and a mixture of the two kinds of covariates.
- estimates and tests average effects for non-orthogonal analysis of variance designs, provided that the covariates are specified as qualitative indicator variables. Other programs typically do not test the average effect at all,or they do not treat the covariates as stochastic regressors, which usually leads to invalid tests of the average effect.
- produces results which are easily interpretable in the analysis of conditional and of average effects (mean differences) between groups.
- estimates and tests *conditional* and *average causal effects*, provided that the covariate-treatment regression is unbiased.

Scope of *EffectLite*

EffectLite ...

- needs Mplus or LISREL in the background
- works with demo version or student version for small models

Probability and Causality

Rolf Steyer
Ivailo Partchev
Ulf Kröhne
Benjamin Nagengast
Christiane Fiege

draft available at www.causal-effects.de

also contains the *Causal Effects Explorer*

and *EffectLite* (with manual).

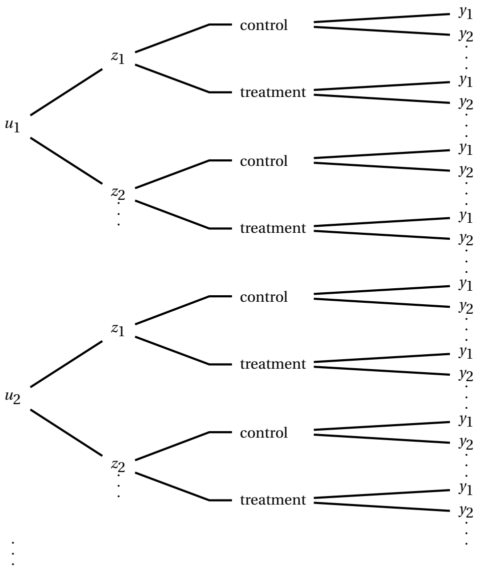
Single-Unit Trials

To which type of *empirical phenomenon* does the theory refer?

- Drawing an observational unit u (e.g., a person) out of a set of units,
- observing the value z of a (possibly multivariate qualitative or quantitative) Z of the unit, if such a covariate is of interest,
- assigning the unit or observing its assignment to one of several experimental conditions (represented by the value x of the treatment variable X),
- recording the numerical value y of the outcome variable Y.

In this single-unit trial, the units u can be considered to be the values of a non-numerical random variable U, and all four random variables, U, Z, X and Y have a joint distribution.

Tree Diagram of a Single-Unit Trial



Introduction: The Problem

- Outside the randomized experiment, i.e., in quasi-experiments, the expected values $E(Y \mid X{=}x)$ of the outcome variable Y given treatment $X{=}x$ and their differences between treatment conditions usually are biased
- The difference $E(Y \mid X{=}x) - E(Y \mid X{=}x')$ can have a different sign than the causal effect (due to selection bias, non-comparable treatment and control groups, etc.)
- If it is not the expected values $E(Y \mid X{=}x)$ and their differences we should estimate and test, what is it that we are after in experimental and quasi-experimental research?
- General answer: Causal effects. But what are causal effects?

Introduction: Rubin's Answer

- Rubin's *individual causal effects*

$$\textit{Rubin's ICE}_{xx'} \equiv Y_x(u) - Y_{x'}(u) \tag{11}$$

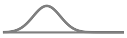
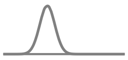
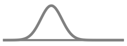
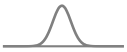
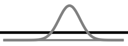
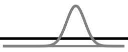
$Y_x(u)$ denotes the potential outcome of unit u under treatment x and $Y_{x'}(u)$ its potential outcome under treatment x'.

- Rubin's *average causal effect*

$$\textit{Rubin's ACE}_{xx'} \equiv \frac{1}{N} \sum_{u=1}^{N} \left[Y_x(u) - Y_{x'}(u) \right] \tag{12}$$

- $Y_x(u)$ and $Y_{x'}(u)$ are fixed numbers. This deterministic outcome assumption ignores the problem of multiple determinacy
- There is even bias at the individual unit level.

Table illustrating Neyman's true yields

Observational-unit variable U (plots)	Covariate 'soil fertility' Z	True yield of variety A $E(Y \mid X{=}0, U{=}u)$	Intra-individual distribution of the outcome variable under control	True yield of variety B $E(Y \mid X{=}1, U{=}u)$	Intra-individual distribution of the outcome variable under treatment	Individual causal effect $E(Y \mid X{=}1, U{=}u) - E(Y \mid X{=}0, U{=}u)$
u_1	low	68		82		14
u_2	low	81		89		8
u_3	low	89		101		12
u_4	low	92		108		16
u_5	high	112		118		6
u_6	high	119		123		4

Introduction: Neyman's Answer

- Neyman's *individual causal effects*

$$\text{Neyman's } ICE_{xx'} \equiv E(Y \mid X{=}x, U{=}u) - E(Y \mid X{=}x', U{=}u)$$

Neyman calls the expected values $E(Y \mid X{=}x, U{=}u)$ "true yields".

- Neyman's *average causal effect*

$$\text{Neyman's } ACE_{xx'} \equiv \frac{1}{N} \sum_{u=1}^{N} \left[E(Y \mid X{=}x, U{=}u) - E(Y \mid X{=}x', U{=}u) \right]$$

- No deterministic outcome assumption, i.e., it is an appropriate answer to the problem of multiple determinacy
- It is not a good answer to the problem of bias at the individual unit level

Stratification or Conditioning

- Both, Neyman and Rubin, base their concepts on stratifying or conditioning on the individual level
- Atomic stratification means stratifying or *conditioning on all potential confounders*, even those which are beyond the individual level
- Hence, we simply replace Neyman's conditional expectation $E_{X=x}(Y \mid U)$ by $E_{X=x}(Y \mid \mathfrak{C}_X)$, where $\mathfrak{C}_X$ is the σ-algebra generated by all potential confounders

Filtration $(\mathfrak{C}_t)_{t\in T}$ and Confounder σ-algebra $\mathfrak{C}_X$

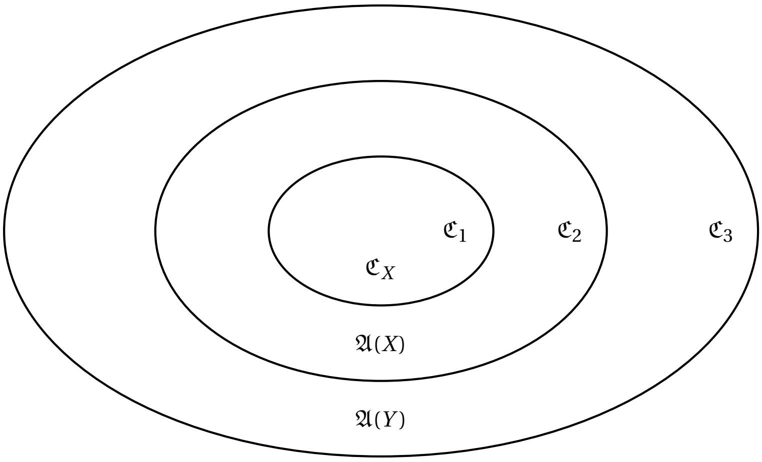


Figure 1: Venn-diagram of a filtration $(\mathfrak{C}_t)_{t\in T}$ with $T = \{1,2,3\}$. $\mathfrak{A}(X)$ denotes the σ-algebra generated by the random variable X.

Probability Space and Causality Space

A *probability space* $(\Omega, \mathfrak{A}, P)$ consists of

- a set Ω of possible outcomes (of the random experiment)
- a σ-algebra $\mathfrak{A}$ of possible events
- a probability measure $P : \Omega \to [0,1]$

A *causality space* $\big((\Omega, \mathfrak{A}, P), X, Y, (\mathfrak{C}_t)_{t\in T}, \mathfrak{C}_X\big)$ consists of:

- a probability space $(\Omega, \mathfrak{A}, P)$
- a random variable X (putative cause)
- a random variable Y (outcome variable)
- a filtration $(\mathfrak{C}_t)_{t\in T}$ (with respect to which random variables and events can be ordered)
- a confounder σ-algebra $\mathfrak{C}_X$ (with respect to which potential confounders are measurable)

True Outcome Variables and True Effect Variables

- Using the confounder σ-algebra $\mathfrak{C}_X$ and the conditional expectation $E_{X=x}(Y \mid \mathfrak{C}_X)$ of Y given $\mathfrak{C}_X$ in treatment x, we can define the *true outcome variables* τ_x and the *true effect variables* $\delta_{xx'} \equiv \tau_x - \tau_{x'}$
- These variables τ_x and $\delta_{xx'}$ are, by definition, purified from confounding
- τ_x and $\delta_{xx'}$ are random variables on the same probability space as the original random variables X and Y. They have expected values, conditional expected values, variances, covariances, etc
- The true outcome variables τ_x play the same role as Rubin's potential outcome variables Y_x.

True Outcomes: Example

Possible outcomes			Random variables								
Units	Treatment	Success	Observational-unit variable U	Treatment variable X	Outcome variable Y	Conditional expected values $E(Y \mid X{=}x, U{=}u)$	Conditional expected values $E(Y \mid X{=}x)$	True outcome variable $\tau_0 = E^{\circ}_{X=0}(Y \mid U)$	True outcome variable $\tau_1 = E^{\circ}_{X=1}(Y \mid U)$	True effect variable $\delta_{10} = \tau_1 - \tau_0$	Probabilities of elementary events $P(\{\omega\})$
Joe	*no*	−	*Joe*	0	0	.70	.60	.70	.80	.10	.144
Joe	*yes*	−	*Joe*	1	0	.80	.42	.70	.80	.10	.004
Joe	*no*	+	*Joe*	0	1	.70	.60	.70	.80	.10	.336
Joe	*yes*	+	*Joe*	1	1	.80	.42	.70	.80	.10	.016
Ann	*no*	−	*Ann*	0	0	.20	.60	.20	.40	.20	.096
Ann	*yes*	−	*Ann*	1	0	.40	.42	.20	.40	.20	.228
Ann	*no*	+	*Ann*	0	1	.20	.60	.20	.40	.20	.024
Ann	*yes*	+	*Ann*	1	1	.40	.42	.20	.40	.20	.152

True Outcomes: Example

Table 3: Simpson's paradox and altered Simpson's paradox: replacement mappings and true outcome variables

Outcomes	Observables			Theoretical random variables									
	Observ.-unit variable U	Treatment variable X	Outcome variable Y	Conditional expectation $E(Y \mid X, U)$	Conditional expectation $E(Y \mid X)$	Replacement mapping r_0	Conditional expectation $E_{X=0}(Y \mid \mathfrak{C}_X)$	Replacement mapping r_1	Conditional expectation $E_{X=1}(Y \mid \mathfrak{C}_X)$	Extension $E^{\circ}_{X=0}(Y \mid \mathfrak{C}_X)$ true outcome variable τ_0	$E(Y \mid X, \mathfrak{C}) \circ r_X$	True effect variable $\delta_{10} \equiv \tau_1 - \tau_0$	Joint probabilities $P(U{=}u, X{=}x, Y{=}y)$
(*J*, *no*, −)	*J*	0	0	.70	.60	(*J*, *no* , −)	.70	(*J*, *yes*, −)	999	.70	.80	.10	.144
(*J*, *yes*, −)	*J*	1	0	.80	.42	(*J*, *no* , −)	999	(*J*, *yes*, −)	.80	.70	.80	.10	.004
(*J*, *no*, +)	*J*	0	1	.70	.60	(*J*, *no*, +)	.70	(*J*, *yes*, +)	999	.70	.80	.10	.336
(*J*, *yes*, +)	*J*	1	1	.80	.42	(*J*, *no* , +)	999	(*J*, *yes*, +)	.80	.70	.80	.10	.016
(*A*, *no*, −)	*A*	0	0	.20	.60	(*A*, *no*, −)	.20	(*A*, *yes*, −)	999	.20	.40	.20	.096
(*A*, *yes*, −)	*A*	1	0	.40	.42	(*A*, *no*, −)	999	(*A*, *yes*, −)	.40	.20	.40	.20	.228
(*A*, *no*, +)	*A*	0	1	.20	.60	(*A*, *no*, +)	.20	(*A*, *yes*, +)	999	.20	.40	.20	.024
(*A*, *yes*, +)	*A*	1	1	.40	.42	(*A*, *no*, +)	999	(*A*, *yes*, +)	.40	.20	.40	.20	.152
Altered Simpson's paradox													
(*J*, *no*, −)	*J*	0	0	.70	.60	(*J*, *no* , −)	.70	(*J*, *yes*, −)	999	.70	999	—	.144
(*J*, *yes*, −)	*J*	1	0	999	.42	(*J*, *no* , −)	999	(*J*, *yes*, −)	999	.70	999	—	.000
(*J*, *no*, +)	*J*	0	1	.70	.60	(*J*, *no*, +)	.70	(*J*, *yes*, +)	999	.70	999	—	.336
(*J*, *yes*, +)	*J*	1	1	999	.42	(*J*, *no* , +)	999	(*J*, *yes*, +)	999	.70	999	—	.000
(*A*, *no*, −)	*A*	0	0	.20	.60	(*A*, *no*, −)	.20	(*A*, *yes*, −)	999	.20	.42	—	.096
(*A*, *yes*, −)	*A*	1	0	.42	.42	(*A*, *no*, −)	999	(*A*, *yes*, −)	.42	.20	.42	—	.232
(*A*, *no*, +)	*A*	0	1	.20	.60	(*A*, *no*, +)	.20	(*A*, *yes*, +)	999	.20	.42	—	.024
(*A*, *yes*, +)	*A*	1	1	.42	.42	(*A*, *no*, +)	999	(*A*, *yes*, +)	.42	.20	.42	—	.168

Example Illustrating the Basic Concepts

Fundamental parameters						Derived parameters				
Unit u	Sampling probability $P(U=u)$	Covariate gender Z	True outcome under control $\tau_0(u) = E(Y \mid X=0, U=u)$	True outcome under treatment $\tau_1(u) = E(Y \mid X=1, U=u)$	Individual treatment probability $P(X=1 \mid U=u)$	Individual causal effect $\delta_{10} = \tau_1(u) - \tau_0(u)$	$P(U=u \mid X=0)$	$P(U=u \mid X=1)$	Extension $E^{\circ}_{X=0}(Y \mid Z)$	Extension $E^{\circ}_{X=1}(Y \mid Z)$
u_1	1/6	m	68	81	3/4	13	1/10	3/14	83	92.5
u_2	1/6	m	78	86	3/4	8	1/10	3/14	83	92.5
u_3	1/6	m	88	100	3/4	12	1/10	3/14	83	92.5
u_4	1/6	m	98	103	3/4	5	1/10	3/14	83	92.5
u_5	1/6	f	106	114	1/4	8	3/10	1/14	111	122
u_6	1/6	f	116	130	1/4	14	3/10	1/14	111	122
Expected values			92.33	102.33	7/12	10			92.33	102.33
$E(Y \mid X=x)$			99.80	96.71						

Average causal effect (*ACE*)	10.000
Prima facie effect (*PFE*)	−3.086

	male	female
Conditional *ACE*s	9.50	11.00
Conditional *PFE*s	9.50	11.00

Basic Concepts of the Theory of Causal Effects

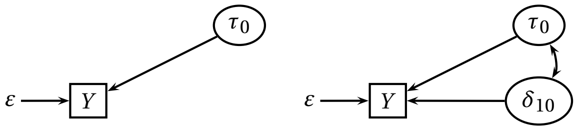
Primitives

$\Omega = \Omega_U \times \Omega_Z \times \Omega_X \times \Omega_O$	The *set of possible outcomes*
$Z_i : \Omega \rightarrow \Omega_{Z_i}$	*Covariates*
$X : \Omega \rightarrow \{0, 1, \ldots, J\}$	*Treatment variable*
$Y : \Omega \rightarrow \mathbb{R}$	*Outcome variable*
$U : \Omega \rightarrow \Omega_U$	*Observational-unit variable*

Theoretical Concepts the Theory of Causal Effects

τ_x	True outcome variables in treatment conditions x
$\delta_{xx'} \equiv \tau_x - \tau_{x'}$	True effect variables
$ACE_{xx'} \equiv E(\delta_{xx'})$	Average causal effect
$ACE_{xx';Z=z} \equiv E(\delta_{xx'} \mid Z=z)$	Average causal effect given $Z=z$
$ACE_{xx';X=x^*} \equiv E(\delta_{xx'} \mid X=x^*)$	Average causal effect given $X=x^*$

Path Diagram



(a) control (b) treatment

$$Y \;=\; \tau_0 + \varepsilon \qquad\qquad Y \;=\; \tau_0 + (\tau_1 - \tau_0) + \varepsilon \;=\; \tau_0 + \delta_{10} + \varepsilon$$

Decomposition of variances
... in control group

$$Var(Y \mid X{=}0) \;=\; Var(\tau_0 \mid X{=}0) \;+\; Var(\varepsilon \mid X{=}0) \tag{13}$$

... in treatment group

$$\begin{aligned} Var(Y \mid X{=}1) \;=\;& Var(\tau_0 \mid X{=}1) \;+\; Var(\varepsilon \mid X{=}1) \\ &+\; Var(\delta_{10} \mid X{=}1) \;+\; 2 \cdot Cov(\tau_0, \delta_{10} \mid X{=}1) \end{aligned} \tag{14}$$

Identification of Average Causal Effects

Core of the Theory

Notation

X — treatment variable Z — covariate
Y — outcome variable U — observational-unit variable

τ_x — true outcome variable
$\delta_{xx'} \equiv \tau_x - \tau_{x'}$ — true causal effect variable
$ACE_{xx'} \equiv E(\delta_{xx'})$ — average causal effect
$ACE_{xx';Z=z} \equiv E(\delta_{xx'} \mid Z{=}z)$ — average causal effect given $Z{=}z$

Unbiasedness

...of the expected value $E(Y \mid X{=}x)$ $:\Leftrightarrow$ $E(Y \mid X{=}x) \;=\; E(\tau_x)$
...of the regression $E_{X=x}(Y \mid Z)$ $:\Leftrightarrow$ $E^{\circ}_{X=x}(Y \mid Z) \;=\; E(\tau_x \mid Z)$

Unbiasedness of the regression $E_{X=x}(Y \mid Z)$ implies:

(1) $E\big[E^{\circ}_{X=x}(Y \mid Z)\big] \;=\; E\big[E(\tau_x \mid Z)\big] \;=\; E(\tau_x)$

Remember : $ACE_{xx'} \equiv E(\tau_x) - E(\tau_{x'})$

and

(2) $E(Y_w \mid X{=}x) \;=\; E(\tau_x)$, with $Y_w \;\equiv\; Y \cdot \left(\sum_{x=0}^{J} I_{X=x} \cdot \frac{P(X{=}x)}{P(X{=}x \mid Z)} \right)$

Let $\mathfrak{C}_X$ denote the confounder σ-algebra.

Each of These Conditions is Sufficient for Unbiasedness

- Z-conditional independence of potential confounders and treatments ($\mathfrak{C}_X \perp X \mid Z$)

$$P(X{=}x \mid \mathfrak{C}_X, Z) \;=\; P(X{=}x \mid Z) \qquad \text{for each value } x \text{ of } X$$

- Homogeneity of the regression ($Y \vdash \mathfrak{C}_X \mid X, Z$)

$$E(Y \mid X, Z, \mathfrak{C}_X) \;=\; E(Y \mid X, Z)$$

- Z-Conditional Strong Causality

$$E(Y \mid X, Z, \mathfrak{C}_X) \;=\; E(Y \mid X, Z) + W, \quad \text{where } W \text{ is } \mathfrak{C}_X\text{-measurable}$$

- Z-conditional independence of true outcomes and treatments ($\tau \perp X \mid Z$ "strong ignorability")

$$P(X{=}x \mid Z, \tau_0, \tau_1, \ldots, \tau_J) \;=\; P(X{=}x \mid Z) \qquad \text{for each value } x \text{ of } X$$

- Z-conditional regressive independence of true outcomes from treatments ($\tau \vdash X \mid Z$)

$$E(\tau_x \mid X, Z) \;=\; E(\tau_x \mid Z) \qquad \text{for each value } x \text{ of } X$$

- Z-unconfoundedness of the regression $E(Y \mid X, Z)$

$$P_{Z=z}(X{=}x \mid \mathfrak{C}_X) \;=\; P_{Z=z}(X{=}x) \qquad \text{or} \qquad E_{X=x,\,Z=z}(Y \mid \mathfrak{C}_X) \;=\; E_{X=x,\,Z=z}(Y)$$

for P_Z-almost all values z of Z and each value x of X

Implication Structure Among Conditions of Unbiasedness

Table 4: Implication Structure Between Causality Conditions

		(iii)	(iv)	(vi)	(vii)
(i)	Z-conditional independence of X and potential confounders ($X \perp \mathfrak{C}_X \mid Z$)	⇒	⇒	⇒	⇒
(ii)	Homogeneity of $E(Y \mid X, Z)$ ($Y \vdash \mathfrak{C}_X \mid X, Z$)	⇒	⇒	⇒	⇒
(iii)	Z-conditional independence of X and true outcomes ($X \perp \tau \mid Z$)		⇒		⇒
(iv)	Z-conditional regressive independence of true outcomes from X ($\tau \vdash X \mid Z$)				⇒
(v)	Z-conditional strong causality of $E(Y \mid X, Z)$			⇒	⇒
(vi)	Z-conditional unconfoundedness of $E(Y \mid X, Z)$				⇒
(vii)	Z-conditional unbiasedness of $E(Y \mid X, Z)$				⇒

Note: ⇒ indicates that condition in row implies condition in column.