



Grundbegriffe der Wahrscheinlichkeitstheorie

- Zufallsexperiment
- Zufallsvariable
- Verteilung
- Erwartungswert
- Varianz, Kovarianz und Korrelation
- Bedingte Erwartung (Regression)



Definition 2.1. Sei A eine Menge von Teilmengen einer Menge Ω . Dann heißt A σ -Algebra, wenn gelten:

- $\Omega \in A$;
- wenn $A \in A$, dann $\bar{A} \in A$ (\bar{A} bezeichnet das Komplement von A);
- wenn A_1, A_2, \dots eine Folge von Elementen aus A ist, dann ist auch deren Vereinigung $A_1 \cup A_2 \cup \dots$ Element von A .



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Expectations, variances, covariances correlation

Expectation

If X takes only the values x_1, \dots, x_n :

$$E(X) = \sum_{i=1}^n x_i \cdot P(X = x_i)$$

Covariance

$$\text{Cov}(X, Y) := E[(X - E(X)) \cdot (Y - E(Y))]$$

Variance

$$\text{Var}(X) := \text{Cov}(X, X) = E[(X - E(X))^2]$$

Standard deviation

$$\text{Std}(X) := +\sqrt{\text{Var}(X)}$$

Correlation

$$\text{Corr}(X, Y) := \begin{cases} \frac{\text{Cov}(X, Y)}{\text{Std}(X) \cdot \text{Std}(Y)}, & \text{if } \text{Std}(X) \text{ and} \\ & \text{Std}(Y) > 0 \\ 0, & \text{otherwise.} \end{cases}$$



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Rules of computation for expectations

Let X and Y with or without indices denote numeric random variables with a joint distribution and finite expectations and let α, β with or without indices denote real-valued constants. Then:

(i) $E(\alpha) = \alpha$

(ii) $E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$



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Rules of computation for variances

Let X and Y with or without indices denote numeric random variables with a joint distribution and finite expectations and let α, β denote real-valued constants. Then:

$$(iii) \quad \text{Var}(X) = E(X^2) - E(X)^2$$

$$(iv) \quad \text{Var}(X) = 0, \quad \text{if } X = \alpha$$

$$(v) \quad \text{Var}(\alpha X) = \alpha^2 \text{Var}(X)$$

$$(vi) \quad \text{Var}(\alpha + X) = \text{Var}(X)$$

$$(vii) \quad \text{Var}(\alpha X + \beta Y) = \alpha^2 \text{Var}(X) + \beta^2 \text{Var}(Y) + 2 \alpha \beta \text{Cov}(X, Y)$$



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Rules of computation for covariances

Let X and Y with or without indices denote numeric random variables with a joint distribution and finite expectations and let α, β (with or without indices) denote real-valued constants. Then:

$$(viii) \quad \text{Cov}(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$$

$$(ix) \quad \text{Cov}(X, Y) = 0, \quad \text{if } X = \alpha$$

$$(x) \quad \text{Cov}(\alpha X, \beta Y) = \alpha \beta \text{Cov}(X, Y)$$

$$(xi) \quad \text{Cov}(\alpha + X, \beta + Y) = \text{Cov}(X, Y)$$

$$(xii) \quad \begin{aligned} \text{Cov}(\alpha_1 X_1 + \alpha_2 X_2, \beta_1 Y_1 + \beta_2 Y_2) \\ = \alpha_1 \beta_1 \text{Cov}(X_1, Y_1) + \alpha_1 \beta_2 \text{Cov}(X_1, Y_2) \\ + \alpha_2 \beta_1 \text{Cov}(X_2, Y_1) + \alpha_2 \beta_2 \text{Cov}(X_2, Y_2) \end{aligned}$$



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Most important definitions concerning conditional expectations and regressions

Conditional expected value
 $E(Y|X = x)$ Let Y be discrete and $P(X = x) > 0$:
 $E(Y|X = x) := \sum_{i=1}^n y_i \cdot P(Y = y_i|X = x)$

Regression or conditional expectation
 $E(Y|X)$
 $E(Y|X_1, \dots, X_m)$ $E(Y|X)$ is the random variable, the values of which are the conditional expected values $E(Y|X = x)$. $E(Y|X)$ is a function of X . Y is called the *Regressand* and X the *Regressor*. X may consist of several random variables X_1, \dots, X_m .

Residual ε $\varepsilon := Y - E(Y|X)$ is the component of Y not determined by $E(Y|X)$.



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Coefficient of determination $R_{Y|X}^2$ $R_{Y|X}^2 := \begin{cases} \frac{\text{Var}[E(Y|X)]}{\text{Var}(Y)}, & \text{if } \text{Var}(Y) > 0, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$

$R_{Y|X}^2$ is the proportion of variance of Y determined by the regression $E(Y|X)$. It indicates the strength of the regressive dependence. It is invariant under one-to-one transformations of X and under linear transformations of Y .

Multiple Correlation The positive square root of the coefficient of determination is called the *multiple correlation*.

Regressive independence $E(Y|X) = E(Y)$



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Rules of computation for regressions

- (i) $E(\alpha | X) = \alpha$
- (ii) $E(\alpha Y_1 + \beta Y_2 | X) = \alpha E(Y_1 | X) + \beta E(Y_2 | X)$
- (iii) $E[E(Y | X)] = E(Y)$
- (iv) $E[f(X) | X] = f(X)$, if $f(X)$ is numeric
- (v) $E[E(Y | X) | f(X)] = E[Y | f(X)]$
- (vi) $E[f(X) Y | X] = f(X) E(Y | X)$, if $f(X)$ is numeric



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Properties of the residual

- (vii) $E(\epsilon) = 0$
- (viii) $Cov[\epsilon, E(Y | X)] = 0$
- (ix) $Var(Y) = Var[E(Y | X)] + Var(\epsilon)$
- (x) $E(\epsilon | X) = 0$
- (xi) $E[\epsilon | f(X)] = 0$
- (xii) $Cov[\epsilon, f(X)] = 0$, if $f(X)$ numeric