Models of Classical Test Theory (CTT)

- Basic concepts of CTT
- Models of CTT
- Implied covariance structure
- Implied mean structure

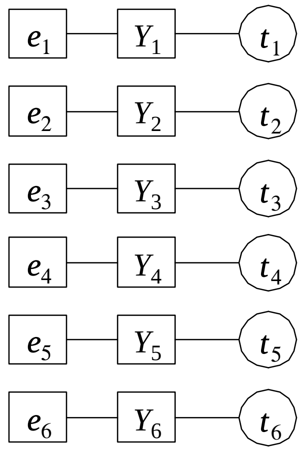
Models of CTT

– Basic Concepts of Classical Test Theory

- Primitives
  - *The set of possible events of the random experiment* $\Omega = \Omega_U \times \Omega_O$
  - *Test Score Variables* $Y_i: \Omega \to \mathrm{IR}$
  - *Projection* $U: \Omega \to \Omega_U$

- Definition of the Theoretical Variables
  - *True Score Variable* $t_i := E(Y_i \mid U)$
  - *Measurement Error Variable* $e_i := Y_i - t_i$

Models of CTT



Models of CTT

– Properties of True Score and Error Variables Implied by Their Definition

- – *Decomposition of the Variables* $Y_i = t_i + e_i$ (1)
- – *Decomposition of the Variances* $Var(Y_i) = Var(t_i) + Var(e_i)$ (2)
- – *Other Properties of True Score and Error Variables implied by their definition*
  $Cov(t_i, e_j) = 0$ (3)
  $E(e_i) = 0$ (4)
  $E(e_i \mid U) = 0$ (5)
- \- *for each (measurable) mapping of U:* $E[e_i \mid f(U)] = 0$ (6)

Models of CTT

– Assumptions and models in CTT

(a_1) t-equivalence $t_i = t_j$

(a_2) essential t-equivalence $t_i = t_j + l_{ij}$, $l_{ij} \in$ IR

(a_3) t-congenerity $t_i = l_{ij0} + l_{ij1}\, t_j$, l_{ij0}, $l_{ij1} \in$ IR, $l_{ij1} > 0$

(b) uncorrelated errors $Cov(e_i, e_j) = 0$, $i \neq j$

(c) equal error variances $Var(e_i) = Var(e_j)$

Models defined by these assumptions

Parallel tests: (a_1), (b) and (c)

Essentially t-equivalent tests: (a_2) and (b)

Congeneric tests: (a_3) and (b)

Models of CTT

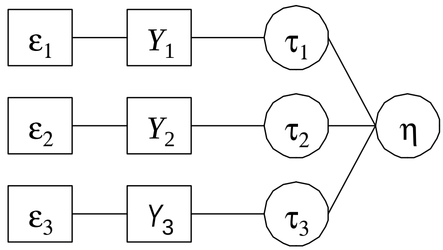
– The Model of Parallel Tests

- Definition Assumptions (a_1), (b) and (c)
- Identification
- $E(\eta) = E(Y_i)$
- $Var(\eta) = Cov(Y_i, Y_j)$, $i \neq j$
- $Var(\varepsilon_i) = Var(Y_i) - Cov(Y_i, Y_j)$, $i \neq j$
- $Rel(Y_i) = Corr(Y_i, Y_j)$, $i \neq j$

metheval
lehrstuhl für methodenlehre
und evaluationsforschung

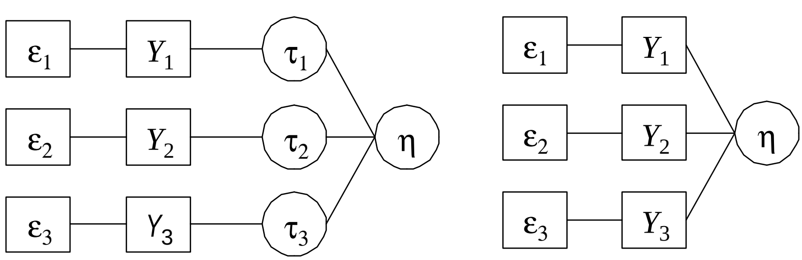
Models of CTT

– The Model of Parallel Tests



Models of CTT

– The Model of Parallel Tests



Models of CTT

– The Model of Parallel Tests

- Implied covariance structure

$$
\begin{aligned}
Cov(Y_1, Y_2) &= Cov(\eta + \varepsilon_1, \eta + \varepsilon_2) \\
&= Cov(\eta, \eta) + Cov(\eta, \varepsilon_2) \\
&\quad + Cov(\varepsilon_1, \eta) + Cov(\varepsilon_1, \varepsilon_2) \\
&= Var(\eta) =: \sigma_\eta^2
\end{aligned}
$$

$$Var(Y_i) - Cov(Y_1, Y_2) = Var(\varepsilon_i) =: \sigma_\varepsilon^2$$

Models of CTT

– The Model of Parallel Tests

- Implied covariance structure

$$
\begin{bmatrix}
\sigma_\eta^2 + \sigma_\varepsilon^2 & \sigma_\eta^2 & \sigma_\eta^2 \\
\sigma_\eta^2 & \sigma_\eta^2 + \sigma_\varepsilon^2 & \sigma_\eta^2 \\
\sigma_\eta^2 & \sigma_\eta^2 & \sigma_\eta^2 + \sigma_\varepsilon^2
\end{bmatrix}
$$

Models of CTT

- The Model of Parallel Tests

- Testability in the total population
- $E(Y_i) = \mu$
  $Var(Y_i) = \sigma_Y^2$
  $Cov(Y_i, Y_j) = s_\eta^2$
-
- Testability within each subpopulation s
- $E^{(s)}(Y_i) = \mu_s$

Models of CTT

– The Model of Essentially τ-Equivalent Tests

- Definition Assumptions (a_2) and (b)
- Fixing the scale of η $E(\eta) = 0$
- Identification
- $Var(\eta) = Cov(Y_i, Y_j), \quad i \neq j$
- $Var(\varepsilon_i) = Var(Y_i) - Cov(Y_i, Y_j), \quad i \neq j$
  $Rel(Y_i) = Cov(Y_i, Y_j) / Var(Y_i), \quad i \neq j$

metheval
lehrstuhl für methodenlehre und evaluationsforschung

Models of CTT

– The model of essentially τ-equivalent tests

- Implied covariance structure

- $$\begin{bmatrix} \sigma_\eta^2 + \sigma_{\varepsilon_1}^2 & \sigma_\eta^2 & \sigma_\eta^2 \\ \sigma_\eta^2 & \sigma_\eta^2 + \sigma_{\varepsilon_2}^2 & \sigma_\eta^2 \\ \sigma_\eta^2 & \sigma_\eta^2 & \sigma_\eta^2 + \sigma_{\varepsilon_3}^2 \end{bmatrix}$$

Models of CTT

– The Model of essentially τ-equivalent tests

- Implied covariance structure

- $$\begin{aligned} Cov(Y_1,Y_2) &= Cov(\eta+\lambda_1+\varepsilon_1, \eta+\lambda_2+\varepsilon_2) \\ &= Cov(\eta,\eta) + Cov(\eta,\varepsilon_2) \\ &\quad + Cov(\varepsilon_1,\eta) + Cov(\varepsilon_1,\varepsilon_2) \\ &= Var(\eta) = \sigma_\eta^2 \end{aligned}$$

$$Var(Y_i) - Cov(Y_1,Y_2) = Var(\varepsilon_i) = \sigma_{\varepsilon_i}^2$$

metheval
lehrstuhl für methodenlehre
und evaluationsforschung

Models of CTT

– The Model of essentially τ-equivalent tests

- Testability
- in the total population
- $Cov(Y_i, Y_j) = \sigma_\eta^2$
- $E(Y_i) - E(Y_j) = E(Y_i - Y_j) = \lambda_{ij}$
-
- in each subpopulation s
- $E^{(s)}(Y_i) - E^{(s)}(Y_j) = E^{(s)}(Y_i - Y_j) = \lambda_{\mathrm{ij}}$

Models of CTT

– The Model of τ-Congeneric Tests

- Definition Assumptions (a_3) and (b)

- Fixing the scale of ***h***
- Either $E(\boldsymbol{h}) = 0$ and $Var(\boldsymbol{h}) = 1$
- or $\lambda_{ij0} = 0$ and $\lambda_{ij1} = 1$.
-

metheval
lehrstuhl für methodenlehre und evaluationsforschung

Table. Illustrating the relationship between manifest and latent variables in the model of τ-congeneric variables

persons	true scores		latent scores	measurements		error scores		
u	τ_1	τ_2	η	Y_1	Y_2	ε_1	ε_2	$P(Y_i = y_i \mid U=u)$
				10	20	-2	-3	1/9
1	12	23	34	12	24	0	1	1/9
				14	25	2	2	1/9
				7	15	-3	-5	1/9
2	10	20	30	9	22	-1	2	1/9
				14	23	4	3	1/9
				3	14	-5	-3	1/9
3	8	17	26	10	15	2	-2	1/9
				11	22	3	5	1/9

Note: Fictitious numbers. Each of the 3 persons has its own (intra-individual) distribution of the *Y*-variables, but only one single score on each of the variables τ_i and η.

The Model of τ-Congeneric Tests

- Identification for fixing $E(h) = 0$ and $Var(h) = 1$.

$$l_{i1} = \sqrt{\frac{Cov(Y_i, Y_j)\ Cov(Y_i, Y_k)}{Cov(Y_j, Y_k)}}, \; i \neq j, i \neq k, j \neq k$$

- $Var(e_i) = Var(Y_i) - \lambda_{i1}^2$

– $Rel(Y_i) = \lambda_{i1}^2 / Var(Y_i)$

Models of CTT

•Testability

- in the total population

$$\frac{Cov(Y_i, Y_k)}{Cov(Y_j, Y_k)} = \frac{Cov(Y_i, Y_l)}{Cov(Y_j, Y_l)}, \quad i \neq k, i \neq l, j \neq k, j \neq l$$

- between subpopulations

$$\frac{E^{(1)}(Y_i) - E^{(2)}(Y_i)}{E^{(1)}(Y_j) - E^{(2)}(Y_j)} = \frac{E^{(3)}(Y_i) - E^{(4)}(Y_i)}{E^{(3)}(Y_j) - E^{(4)}(Y_j)}$$

Models of CTT

- The model of congeneric tests
- Implied covariance structure

$$Cov(Y_i, Y_j) = Cov(\lambda_{i1}\eta + \varepsilon_i, \lambda_{j1}\eta + \varepsilon_j)$$
$$= \lambda_{i1}\lambda_{j2}\, Var(\eta)$$

$$Var(Y_i) = Var(\lambda_{i1}\eta + \varepsilon_i)$$
$$= \lambda_{i1}^{\ 2}\, Var(\eta) + Var(\varepsilon_i)$$

metheval
lehrstuhl für methodenlehre und evaluationsforschung

Models of CTT

$$
\begin{bmatrix}
\lambda_{11}{}^{2}\,\sigma_{\eta}{}^{2}+\sigma_{\varepsilon_1}{}^{2} & & \\
\lambda_{11}\lambda_{12}\,\sigma_{\eta}{}^{2} & \lambda_{12}{}^{2}\,\sigma_{\tau}{}^{2}+\sigma_{\varepsilon_2}{}^{2} & \\
\lambda_{11}\lambda_{13}\,\sigma_{\eta}{}^{2} & \lambda_{12}\lambda_{13}\,\sigma_{\eta}{}^{2} & \lambda_{13}{}^{2}\,\sigma_{\eta}{}^{2}+\sigma_{\varepsilon_3}{}^{2}
\end{bmatrix}
$$

Models of CTT

$$
\frac{Cov(Y_1,Y_2)\cdot Cov(Y_3,Y_1)}{Cov(Y_2,Y_3)}=\frac{\lambda_{11}\lambda_{12}\cdot\lambda_{11}\lambda_{13}}{\lambda_{12}\lambda_{13}}
$$

$$
=\lambda_{11}{}^{2}
$$

metheval
lehrstuhl für methodenlehre
und evaluationsforschung