Models of Classical Test Theory (CTT)

- Basic concepts of CTT
- Models of CTT
- Implied covariance structure
- Implied mean structure

Models of Classical Test Theory (CTT)

Based on:

Steyer, R. und Eid, M. (2002). *Messen und Testen. (2. Auflage.)* Berlin: Springer.

Steyer, R. (1989). Models of Classical Psychometric Test Theory as Stochastic Measurement Models: Representation, Uniqueness, Meaningfulness, Identifiability, and Testability. *Methodika, 3,* 25-60.

Steyer, R. (2001). Classical Test Theory. In: Ragin, C. and Cook, T. (Eds.), International Encyclopedia of the Social and Behavioural Sciences. Logic of Inquiry and Research Design (pp. 1955-1962). Pergamon, Oxford.

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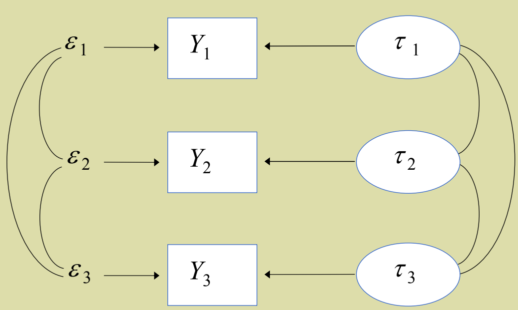
Models of CTT

Basic Concepts of Classical Test Theory

- Primitives
  - *The set of possible events of the random experiment* $\Omega = \Omega_U \times \Omega_O$
  - *Test Score Variables* $Y_i: \Omega \rightarrow \mathrm{IR}$
  - *Projection* $U: \Omega \rightarrow \Omega_U$
- Definition of the Theoretical Variables
  - *True Score Variable* $\tau_i := E(Y_i \mid U)$
  - *Measurement Error Variable* $\varepsilon_i := Y_i - \tau_i$

Models of CTT



Models of CTT

Properties of True Score and Error Variables Implied by Their Definition

- *Decomposition of the Variables* $Y_i = \tau_i + \varepsilon_i$ (1)
- *Decomposition of the Variances* $Var(Y_i) = Var(\tau_i) + Var(\varepsilon_i)$ (2)
- *Other Properties of True Score and Error Variables implied by their definition* $Cov(\tau_i, \varepsilon_j) = 0$ (3)
  $E(\varepsilon_i) = 0$ (4)
  $E(\varepsilon_i \mid U) = 0$ (5)
- *for each (measurable) mapping of U:* $E[\varepsilon_i \mid f(U)] = 0$ (6)

Models of CTT

Assumptions and models in CTT

(a_1) *τ-equivalence* $\tau_i = \tau_j$
(a_2) *essential τ-equivalence* $\tau_i = \tau_j + \lambda_{ij}$, $\lambda_{ij} \in \mathrm{IR}$
(a_3) *τ-congenerity* $\tau_i = \lambda_{ij0} + \lambda_{ij1}\,\tau_j$, $\lambda_{ij0}, \lambda_{ij1} \in \mathrm{IR}$, $\lambda_{ij1} > 0$
(b) *uncorrelated errors* $Cov(\varepsilon_i, \varepsilon_j) = 0$, $i \neq j$
(c) *equal error variances* $Var(\varepsilon_i) = Var(\varepsilon_j)$

Models defined by these assumptions
Parallel tests: (a_1), *(b) and (c)*
Essentially τ-equivalent tests: (a_2) *and (b)*
Congeneric tests: (a_3) *and (b)*

Models of CTT

The Model of Parallel Tests

Definition: Assumptions (a_1), (b) and (c)

(a_1) *τ-equivalence* $\tau_i = \tau_j$

(b) *uncorrelated errors* $Cov(\varepsilon_i, \varepsilon_j) = 0, \quad i \neq j$

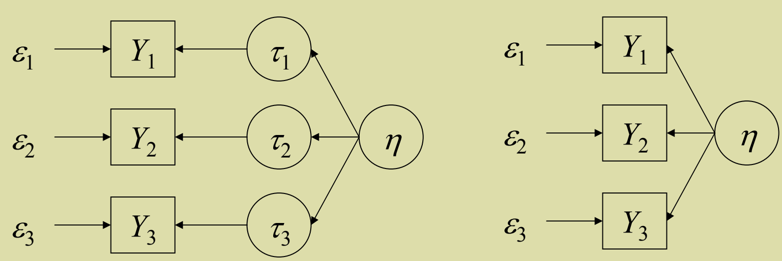
(c) *equal error variances* $Var(\varepsilon_i) = Var(\varepsilon_j)$

Identification:

- $E(\eta) = E(Y_i)$
- $Var(\eta) = Cov(Y_i, Y_j), \quad i \neq j$
- $Var(\varepsilon_i) = Var(Y_i) - Cov(Y_i, Y_j), \quad i \neq j$
- $Rel(Y_i) = Corr(Y_i, Y_j), \quad i \neq j$

Models of CTT

The Model of Parallel Tests



Models of CTT

The Model of Parallel Tests

Implied covariance structure

$$
\begin{aligned}
Cov(Y_1, Y_2) &= Cov(\eta + \varepsilon_1,\ \eta + \varepsilon_2) \\
&= Cov(\eta, \eta) + Cov(\eta, \varepsilon_2) \\
&\quad + Cov(\varepsilon_1, \eta) + Cov(\varepsilon_1, \varepsilon_2) \\
&= Var(\eta) =: \sigma_\eta{}^2 \\
Var(Y_i) &= Var(\eta + \varepsilon_i) \\
&= Var(\eta) + Var(\varepsilon_i) + 2 \cdot Cov(\eta, \varepsilon_i) \\
&= \sigma_\eta{}^2 + \sigma_\varepsilon{}^2
\end{aligned}
$$

Models of CTT

The Model of Parallel Tests

Implied covariance matrix

$$
\begin{bmatrix}
\sigma_\eta{}^2 + \sigma_\varepsilon{}^2 & \sigma_\eta{}^2 & \sigma_\eta{}^2 \\
\sigma_\eta{}^2 & \sigma_\eta{}^2 + \sigma_\varepsilon{}^2 & \sigma_\eta{}^2 \\
\sigma_\eta{}^2 & \sigma_\eta{}^2 & \sigma_\eta{}^2 + \sigma_\varepsilon{}^2
\end{bmatrix}
$$

Models of CTT

The Model of Parallel Tests

Testability in the total population

$E(Y_i) = \mu$
$Var(Y_i) = \sigma_\eta^2 + \sigma_\varepsilon^2 = \sigma_Y^2$
$Cov(Y_i, Y_j) = \sigma_\eta^2, \quad i \neq j$

Testability within each subpopulation s

$E^{(s)}(Y_i) = \mu_s$

Models of CTT

The Model of Essentially τ-Equivalent Tests

Definition: Assumptions (a_2) and (b)

(a_2) *essential τ-equivalence* $\tau_i = \tau_j + \lambda_{ij}, \quad \lambda_{ij} \in$ IR
(b) *uncorrelated errors* $Cov(\varepsilon_i, \varepsilon_j) = 0, \quad i \neq j$

Fixing the scale of η: $E(\eta) = 0$

Identification:

$Var(\eta) = Cov(Y_i, Y_j), \quad i \neq j$
$Var(\varepsilon_i) = Var(Y_i) - Cov(Y_i, Y_j), \quad i \neq j$
$Rel(Y_i) = Cov(Y_i, Y_j) / Var(Y_i), \quad i \neq j$

Models of CTT

The model of essentially τ-equivalent tests

Implied covariance matrix

$$\begin{bmatrix} \sigma_\eta{}^2 + \sigma_{\varepsilon_1}{}^2 & \sigma_\eta{}^2 & \sigma_\eta{}^2 \\ \sigma_\eta{}^2 & \sigma_\eta{}^2 + \sigma_{\varepsilon_2}{}^2 & \sigma_\eta{}^2 \\ \sigma_\eta{}^2 & \sigma_\eta{}^2 & \sigma_\eta{}^2 + \sigma_{\varepsilon_3}{}^2 \end{bmatrix}$$

Models of CTT

The Model of essentially τ-equivalent tests

Implied covariance structure

$$\begin{aligned} Cov(Y_1, Y_2) &= Cov(\eta + \lambda_1 + \varepsilon_1, \eta + \lambda_2 + \varepsilon_2) \\ &= Cov(\eta, \eta) + Cov(\eta, \varepsilon_2) \\ &\quad + Cov(\varepsilon_1, \eta) + Cov(\varepsilon_1, \varepsilon_2) \\ &= Var(\eta) = \sigma_\eta{}^2 \\ Var(Y_i) &= Var(\eta + \lambda_i + \varepsilon_i) \\ &= Var(\eta) + Var(\varepsilon_i) + 2 \cdot Cov(\eta, \varepsilon_i) \\ &= \sigma_\eta{}^2 + \sigma_{\varepsilon_i}{}^2 \end{aligned}$$

Models of CTT

The Model of essentially τ-equivalent tests

Testability

in the total population

$$Cov(Y_i, Y_j) = \sigma_\eta^2, \quad i \neq j$$

$$E(Y_i) - E(Y_j) = E(Y_i - Y_j) = \lambda_{ij}$$

in each subpopulation *s*

$$E^{(s)}(Y_i) - E^{(s)}(Y_j) = E^{(s)}(Y_i - Y_j) = \lambda_{ij}$$

Models of CTT

The Model of τ-Congeneric Tests

Definition: Assumptions (a_3) and (b)

(a_3) *τ-congenerity* $\tau_i = \lambda_{ij0} + \lambda_{ij1}\,\tau_j, \quad \lambda_{ij0}, \lambda_{ij1} \in \mathrm{IR}, \quad \lambda_{ij1} > 0$

(b) *uncorrelated errors* $Cov(\varepsilon_i, \varepsilon_j) = 0, \quad i \neq j$

Fixing the scale of η

Either $E(\eta) = 0$ and $Var(\eta) = 1$

or $\lambda_{ij0} = 0$ and $\lambda_{ij1} = 1$.

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Table. Illustrating the relationship between manifest and latent variables in the model of τ-congeneric variables

persons	true scores		latent scores	measurements		error scores		
u	τ_1	τ_2	η	Y_1	Y_2	ε_1	ε_2	$P(Y_i = y_i \mid U=u)$
				10	20	-2	-3	1/3
1	12	23	34	12	24	0	1	1/3
				14	25	2	2	1/3
				7	15	-3	-5	1/3
2	10	20	30	9	22	-1	2	1/3
				14	23	4	3	1/3
				3	14	-5	-3	1/3
3	8	17	26	10	15	2	-2	1/3
				11	22	3	5	1/3

Note: Fictitious numbers. Each of the 3 persons has its own (intra-individual) distribution of the Y-variables, but only one single score on each of the variables τ_i and η.

The Model of τ-Congeneric Tests

Identification for fixing $E(\eta) = 0$ and $Var(\eta) = 1$.

$$\lambda_{i1} = \sqrt{\frac{Cov(Y_i, Y_j)\; Cov(Y_i, Y_k)}{Cov(Y_j, Y_k)}}, \; i \neq j, i \neq k, j \neq k$$

$$Var(\varepsilon_i) = Var(Y_i) - \lambda_{i1}^2$$

$$Rel(Y_i) = \lambda_{i1}^2 \,/\, Var(Y_i)$$

Models of CTT

Identification of the loadings
(for fixing $E(\eta) = 0$ and $Var(\eta) = 1$).

$$\frac{Cov(Y_1, Y_2) \cdot Cov(Y_3, Y_1)}{Cov(Y_2, Y_3)} = \frac{\lambda_{11}\lambda_{12} \cdot \lambda_{11}\lambda_{13}}{\lambda_{12}\lambda_{13}}$$

$$= \lambda_{11}{}^2$$

Models of CTT

Testability

in the total population

$$\frac{Cov(Y_i, Y_k)}{Cov(Y_j, Y_k)} = \frac{Cov(Y_i, Y_l)}{Cov(Y_j, Y_l)}, \quad i \neq k, i \neq l, j \neq k, j \neq l$$

between subpopulations

$$\frac{E^{(1)}(Y_i) - E^{(2)}(Y_i)}{E^{(1)}(Y_j) - E^{(2)}(Y_j)} = \frac{E^{(3)}(Y_i) - E^{(4)}(Y_i)}{E^{(3)}(Y_j) - E^{(4)}(Y_j)}$$

Models of CTT

The model of congeneric tests

implied covariance structure

$$Cov(Y_i, Y_j) = Cov(\lambda_{i1}\eta + \varepsilon_i, \lambda_{j1}\eta + \varepsilon_j)$$
$$= \lambda_{i1}\lambda_{j2}\, Var(\eta), \quad i \neq j$$

$$Var(Y_i) = Var(\lambda_{i1}\eta + \varepsilon_i)$$
$$= \lambda_{i1}^{\;2}\, Var(\eta) + Var(\varepsilon_i)$$

Models of CTT

Implied covariance matrix

$$\begin{bmatrix} \lambda_{11}^{\;2}\,\sigma_\eta^{\;2} + \sigma_{\varepsilon_1}^{\;2} & & \\ \lambda_{11}\lambda_{12}\,\sigma_\eta^{\;2} & \lambda_{12}^{\;2}\,\sigma_\eta^{\;2} + \sigma_{\varepsilon_2}^{\;2} & \\ \lambda_{11}\lambda_{13}\,\sigma_\eta^{\;2} & \lambda_{12}\lambda_{13}\,\sigma_\eta^{\;2} & \lambda_{13}^{\;2}\,\sigma_\eta^{\;2} + \sigma_{\varepsilon_3}^{\;2} \end{bmatrix}$$