Basic concepts of probability theory

Basic concepts of probability theory

Based on:

Steyer, R. (1988). Conditional Expectations: An Introduction to the Concept and its Applications in Empirical Sciences. *Methodika, 2,* 53-78.

Steyer, R. (2003). *Wahrscheinlichkeit und Regression.* Berlin: Springer.

Basic concepts of probability theory

- Random experiment
- Random variable
- Distribution
- Expected value
- Variance, Covariance and Correlation
- Conditional expectation (Regression)

Basic concepts of probability theory

Expectations, variances, covariances correlation

Expectation — If X takes only the values $x_1, \ldots, x_n$:

$$E(X) = \sum_{i=1}^{n} x_i \cdot P(X = x_i)$$

Covariance

$$Cov(X, Y) := E\left[\,[X - E(X)] \cdot [Y - E(Y)]\,\right]$$

Variance

$$Var(X) := Cov(X, X) = E\left[\,[X - E(X)]^2\right]$$

Standard deviation

$$Std(X) := +\sqrt{Var(X)}$$

Correlation

$$Corr(X,Y) := \begin{cases} \dfrac{Cov(X,Y)}{Std(X) \cdot Std(Y)}, & \text{if } Std(X) \text{ and } Std(Y) > 0 \\ 0, & otherwise. \end{cases}$$

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Rules of computation for expectations

Let X and Y with or without indices denote numeric random variables with a joint distribution and finite expectations and let α, β with or without indices denote real-valued constants. Then:

(i) $E(\alpha) = \alpha$

(ii) $E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$

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Rules of computation for variances

Let X and Y with or without indices denote numeric random variables with a joint distribution and finite expectations and let α, β denote real-valued constants. Then:

(iii) $Var(X) = E(X^2) - E(X)^2$

(iv) $Var(X) = 0$, if $X = \alpha$

(v) $Var(\alpha X) = \alpha^2 Var(X)$

(vi) $Var(\alpha + X) = Var(X)$

(vii) $Var(\alpha X + \beta Y) = \alpha^2 Var(X) + \beta^2 Var(Y) + 2 \alpha \beta Cov(X, Y)$

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Rules of computation for covariances

Let X and Y with or without indices denote numeric random variables with a joint distribution and finite expectations and let α, β (with or without indices) denote real-valued constants. Then:

(viii) $Cov\,(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

(ix) $Cov\,(X, Y) = 0, \quad \text{if } X = \alpha$

(x) $Cov\,(\alpha X,\ \beta Y) = \alpha\, \beta\, Cov\,(X, Y)$

(xi) $Cov\,(\alpha + X, \beta + Y) = Cov\,(X, Y)$

(xii) $Cov(\alpha_1 X_1 + \alpha_2 X_2,\ \beta_1 Y_1 + \beta_2 Y_2)$
$= \alpha_1 \beta_1\, Cov(X_1, Y_1) + \alpha_1 \beta_2\, Cov(X_1, Y_2)$
$+ \alpha_2 \beta_1\, Cov(X_2, Y_1) + \alpha_2 \beta_2\, Cov(X_2, Y_2)$

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Most important definitions concerning conditional expectations and regressions

Conditional expected value $E(Y \mid X = x)$	Let Y be discrete and $P(X = x) > 0$: $E(Y \mid X = x) := \sum_{i=1}^{n} y_i \cdot P(Y = y_i \mid X = x)$
Regression or conditional expectation $E(Y \mid X)$ $E(Y \mid X_1, ..., X_m)$	$E(Y \mid X)$ is the random variable, the values of which are the conditional expected values $E(Y \mid X = x)$. $E(Y \mid X)$ is a function of X. Y is called the *Regressand* and X the *Regressor*. X may consist of several random variables $X_1, ..., X_m$.
Residual ε	$\varepsilon := Y - E(Y \mid X)$ is the component of Y not determined by $E(Y \mid X)$.

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Coefficient of determination $R^2_{Y|X}$

$$R^2_{Y|X} := \begin{cases} \frac{Var[E(Y|X)]}{Var(Y)}, & \text{if } Var(Y) > 0, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

$R^2_{Y|X}$ is the proportion of variance of Y determined by the by the regression $E(Y|X)$. It indicates the strength of the regressive dependence. It is invariant under one-to-one transformations of X and under linear transformations of Y.

Multiple Correlation $R_{Y|X}$

The positive square root of the coefficient of determination is called the *multiple correlation*.

Regressive independence

$E(Y|X) = E(Y)$

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Rules of computation for regressions

(i) $E(\alpha \mid X) = \alpha$

(ii) $E(\alpha\, Y_1 + \beta\, Y_2 \mid X) = \alpha\, E(Y_1 \mid X) + \beta\, E(Y_2 \mid X)$

(iii) $E[E(Y \mid X)] = E(Y)$

(iv) $E[f(X) \mid X] = f(X)$, if $f(X)$ is numeric

(v) $E[E(Y \mid X) \mid f(X)] = E[Y \mid f(X)]$

(vi) $E[f(X) \cdot Y \mid X] = f(X) \cdot E(Y \mid X)$, if $f(X)$ is numeric

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Properties of the residual

(vii) $E(\varepsilon) = 0$

(viii) $Cov[\varepsilon, E(Y \mid X)] = 0$

(ix) $Var(Y) = Var[E(Y \mid X)] + Var(\varepsilon)$

(x) $E(\varepsilon \mid X) = 0$

(xi) $E[\varepsilon \mid f(X)] = 0$

(xii) $Cov[\varepsilon, f(X)] = 0$, if $f(X)$ numeric