Probability and Fundamentals of Inferential Statistics

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Literature

- Eid, M., Gollwitzer, M. & Schmitt, M. (2005). *Statistik und Forschungsmethoden.* Weinheim: Beltz.
- Hays, W. L. (1994). *Statistics.* Fort Worth, TX: Harcourt Brace College Publishers.
- Holling, H. & Gediga, G. (2013). *Statistik: Wahrscheinlichkeitstheorie und Schätzverfahren.* Göttingen: Hogrefe.
- Müller, P. H. (Hrsg.) (1975). *Lexikon der Stochastik.* Berlin: Akademie-Verlag.
- Sachs, L. & Hedderich, J. (2006). *Angewandte Statistik. Methodensammlung mit R.* Berlin: Springer.
- **Steyer, R. (2003).** ***Wahrscheinlichkeit und Regression.*** **Berlin: Springer.**
- **Steyer, R. & Nagel, W. (in press).** ***Probability and Conditional Expectation. Fundamentals for the Empirical Sciences.*** **Chichester: Wiley.**
- Walz, G. (2004). *Lexikon der Statistik.* München: Elsevier.

Why probability theory and inferential statistics?

- Probability theory is the language in which we can formulate precisely our theories and hypotheses so that we can test them empirically
- Probability theory is the language in which can develop the techniques of inferential statistics.
- The techniques of inferential statistics help us to find out if our theories and hypotheses are consistent with empirical observations

The lecture has two parts

- Probability theory
- Fundamentals of inferential statistics

The probability space and its components

A fundamental concept of probability theory is *probability space.* In many applications, such a probability space is the mathematical representation of the kind of empirical phenomenon that we are interested in empirical sciences: a *random experiment.*

A probability space consists of the following components:

- the set of the (possible) outcomes of the random experiment, often denoted Ω
- the set of (possible) events, often denoted $\mathscr{A}$
- the probability measure, often denoted P.

The triple $(\Omega, \mathscr{A}, P)$ is called a *probability space.*

These components are now treated in more detail.

Examples: The set of (all possible) outcomes of the random experiment

Examples of a set of (all possible) outcomes

- The set of all possible outcomes of *flipping a coin* is $\Omega_1 = \{h,\ t\}$.
- The set of all possible outcomes of *tossing a dice* is $\Omega_2 = \{\omega_1, \ldots, \omega_6\}$, where ω_1 denotes tossing a "one" and ω_6 denotes tossing a "six".
- The set of all possible outcomes of *flipping two coins* is $\Omega_3 = \Omega_1 \times \Omega_1 = \{(h,h),\ (h,t),\ (t,h),\ (t,t)\}$. This set contains 4 elements, the pairs (h,h), (h,t), (t,h), and (t,t). The pair (h,t) denotes the outcome "heads" for coin 1 and "tails" for coin 2.

Example: Joe and Ann with all possible outcomes

We consider the following random experiment:

- sample a person from the set $\Omega = \{Joe, Ann\}$ of persons
- observe whether or not a person is treated
- observe whether or not a success criterion is reached

Table 1: Joe and Ann with all possible outcomes

Elements of Ω		
Unit	Treatment	Success
(Joe,	no,	$-$)
(Joe,	no,	$+$)
(Joe,	yes,	$-$)
(Joe,	yes,	$+$)
(Ann,	no,	$-$)
(Ann,	no,	$+$)
(Ann,	yes,	$-$)
(Ann,	yes,	$+$)

The set of (all possible) events

The set of all possible events of a random experiments must be a σ-*algebra*.

Definition

Let $\mathscr{A}$ be a subset of a nonempty set Ω and let $A^c := \Omega \setminus A := \{\omega \in \Omega : \omega \notin A\}$ denote the *complement* of A. If

(a) $\Omega \in \mathscr{A}$
(b) if $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$
(c) if $A_1, A_2, \ldots \in \mathscr{A}$, then $A_1 \cup A_2 \cup \ldots \in \mathscr{A}$.

then $\mathscr{A}$ is called a σ-*algebra* (or a σ-*field*) on Ω.

Events are subsets of the set of all possible outcomes.

First examples of a σ-algebra

- The power set $\mathscr{P}(\Omega)$, i.e., the set of all subsets of Ω, is a σ-*algebra* on Ω, provided that Ω is a nonempty set.
- $\{\emptyset, \Omega\}$ is a σ-*algebra* on Ω.
- $\{\emptyset, \Omega, A, A^c\}$ is a σ-*algebra* on Ω, provided that Ω is a nonempty set.

An event, consisting of just one single outcome, is called an *elementary event*.

Another example of a σ-algebra

The set of all possible events when *tossing* 2 *dices* is $\Omega = \{(h,h),(h,t),(t,h),(t,t)\}$.

The set $\mathscr{A} = \mathscr{P}(\Omega)$ is the set of all subsets of Ω. It contains $2^4 = 16$ elements:

$$\begin{aligned}\mathscr{A} := \Big\{ & \emptyset,\ \Omega,\ \{(h,h)\},\ \{(h,t)\},\ \{(t,h)\},\ \{(t,t)\}\ , \\ & \{(h,h),\ (h,t)\},\ \{(h,h),\ (t,h)\},\ \{(h,h),\ (t,t)\}\ , \\ & \{(h,t),\ (t,h)\},\ \{(h,t),\ (t,t)\},\ \{(t,h),\ (t,t)\}\ , \\ & \{(h,h),\ (h,t),\ (t,h)\},\ \{(h,h),\ (t,t),\ (t,t)\}\ , \\ & \{(h,h),\ (t,h),\ (t,t)\},\ \{(h,t),\ (t,h),\ (t,t)\}\ \Big\}\end{aligned}$$

Example: Joe and Ann with some σ-algebras

Let $\Omega = \{(Joe, no, \text{-}), (Joe, no, +), (Joe, yes, \text{-}), (Joe, yes, +),$
$(Ann, no, \text{-}), (Ann, no, +), (Ann, yes, \text{-}), (Ann, yes, +)\}$

- $\mathscr{A}_1 = \mathscr{P}(\Omega)$. This σ-algebra contains $2^8 = 256$ events
- $\mathscr{A}_2 = \{\emptyset, \Omega, A, A^c\}$ with
  $A = \{(Joe, no, \text{-}), (Joe, no, +), (Joe, yes, \text{-}), (Joe, yes, +)\}$
  and
  $A^c = \{(Ann, no, \text{-}), (Ann, no, +), (Ann, yes, \text{-}), (Ann, yes, +)\}$. This σ-algebra contains 4 elements
- $\mathscr{A}_3 = \{\emptyset, \Omega, B, B^c\}$ with
  $B = \{(Joe, no, \text{-}), (Joe, no, +), (Ann, no, \text{-}), (Ann, no, +)\}$
  and
  $B^c = \{(Joe, yes, \text{-}), (Joe, yes, +), (Ann, yes, \text{-}), (Ann, yes, +)\}$. This σ-algebra also contains 4 elements

Definitions of a probability measure and a probability space

Definition

Let $\mathscr{A}$ be σ-*algebra* on a nonempty set Ω, and let $[0,1]$ be the interval of all real numbers between 0 and 1, inclusively. A function $P: \mathscr{A} \to [0,1]$ is called a *probability measure on* $\mathscr{A}$, if:

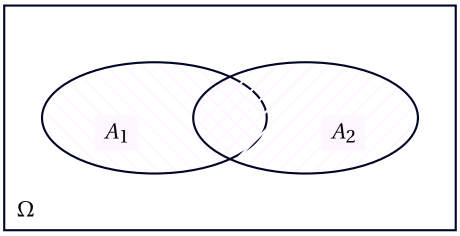
(a) $P(A) \geq 0$, for all $A \in \mathscr{A}$ (*nonnegativity*)
(b) If $A_1, A_2, \ldots \in \mathscr{A}$ and all A_i are pairwise disjoint (i.e., $A_i \cap A_j = \emptyset$, if $i \neq j$, $i, j = 1, 2, \ldots$), then
$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$ (σ-*additivity*)
(c) $P(\Omega) = 1$ (*standardization*)

The triple $(\Omega, \mathscr{A}, P)$ is called a *probability space* if Ω is a nonempty set, $\mathscr{A}$ a σ-algebra on Ω, and P a probability measure on $\mathscr{A}$.

Rules of computation for probabilities

Let $(\Omega, \mathscr{A}, P)$ be a probability space, A, A_1, $A_2 \subset \Omega$, and let $A_1 \setminus A_2 := \{\omega \in \Omega : \omega \in A_1 \text{ and } \omega \notin A_2\}$ denote the *set difference*.

- If $A_2 \subset A_1$, then $P(A_1 \setminus A_2) = P(A_1) - P(A_2)$ and $P(A_1) \geq P(A_2)$
- $P(A_1 \setminus A_2) = P(A_1) - P(A_1 \cap A_2)$
- $P(A^c) = 1 - P(A)$
- $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$



Example of a probability measure

The set of all possible outcomes of flipping two coins is
$\Omega = \{(h,h),(h,t),(t,h),(t,t)\}$. The power set is a σ-algebra $\mathscr{A}$ mit $2^4 = 16$ elements.

The probability measure $P : \mathscr{A} \to [0,1]$ has the following values:

$P(\{(h,h)\}) = P(\{(h,t)\}) = P(\{(t,h)\}) = P(\{(t,t)\}) = 1/4$

The probabilities of all other events can be computed from these probabilities using condition (b) of the definition of a probability measure.

$P(\{(h,h),(h,t)\}) = 1/4 + 1/4 = 1/2.$
$P(\{(h,h),(t,h)\}) = 1/4 + 1/4 = 1/2$

The probability of an event such as $\{(h,h),(h,t),(t,h)\}$ can be computed as follows:

$$P(\{(h,h),(h,t),(t,h)\}) = P(\{(h,h),(h,t)\}) + P(\{(t,h)\}) = 1/2 + 1/4 = 3/4,$$

because the events $\{(h,h),(h,t)\}$ and $\{(t,h)\}$ are disjoint.

Example: Joe and Ann with probabilities for elementary events

Table 2: Joe and Ann with probabilities for elementary events

Elements of Ω			
Unit	Treatment	Success	$P(\{\omega\})$
(*Joe*,	*no*,	−)	.09
(*Joe*,	*no*,	+)	.21
(*Joe*,	*yes*,	−)	.04
(*Joe*,	*yes*,	+)	.16
(*Ann*,	*no*,	−)	.24
(*Ann*,	*no*,	+)	.06
(*Ann*,	*yes*,	−)	.16
(*Ann*,	*yes*,	+)	.04

- Which is the event that Joe is drawn and which is the probability of this event?
- Which is the event that Joe is sampled, treated, and successful, and which is the probability of this event?
- Which is the event that Joe is sampled, treated, and not successful, and which is the probability of this event?

Read more and find exercises

Steyer, R. (2003). *Wahrscheinlichkeit und Regression.* Berlin: Springer.

Steyer, R. & Nagel, W. (in press). *Probability and Conditional Expectation. Fundamentals for the Empirical Sciences.* Chichester: Wiley.