Probability and Inferential Statistis

SS 16

Random Variables

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Random Variable

Preliminary considerations

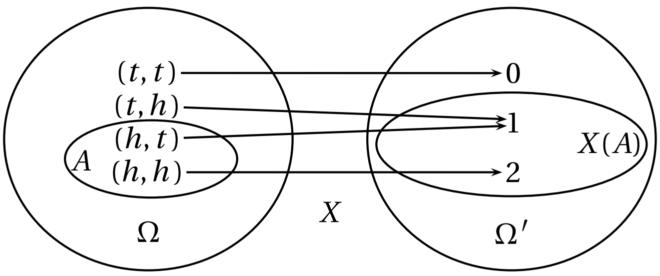
A *random variable* X on a probability space $(\Omega, \mathscr{A}, P)$ assigns one and only one value x to each outcome $\omega \in \Omega$ of the random experiment represented by $(\Omega, \mathscr{A}, P)$. These values can be numbers, but they can also be elements of any other set. Hence, a random variable is a mapping. This is denoted by $X : \Omega \to \Omega'$, where Ω is the set of possible outcomes and Ω' the set of possible values of X. The definition of a random variable also requires that there is a σ-algebra $\mathscr{A}'$ on Ω'.

A random variable is defined such that it has a *distribution*. Different random variables can be stochastically *dependent* or *independent* of each other.

Number of heads when tossing two coins: Image

Consider flipping two coins. Then the set of all possible outcomes is

$$\Omega = \Big\{(h, h), (h, t), (t, h), (t, t)\Big\}.$$



An event A and its image $X(A)$ under the random variable X ("number of heads")

Inverse Image

In the definition of a random variable $X : \Omega \to \Omega'$ we use the concept of an inverse image. The *inverse image* $X^{-1}(A')$ of the set A' under X is the event that X takes on a value in the set $A' \subset \Omega'$, i.e.,

$$X^{-1}(A') := \{\omega \in \Omega : X(\omega) \in A'\}.$$

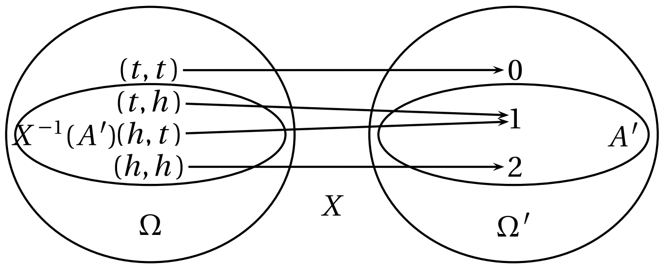
A value x of a random variable X represents the event $X^{-1}(\{x\})$ that X takes on the value x. This event is also denoted by $\{X=x\}$, i.e.,

$$\{X=x\} := X^{-1}(\{x\}).$$

Number of heads when tossing two coins: Inverse image

The set of all possible outcomes of flipping two coins is

$$\Omega = \{(h,h),(h,t),(t,h),(t,t)\}.$$



A set A' and its inverse image $X^{-1}(A')$ under the random variable X ("number of heads")

Notation of Events Represented by a Random Variable

The set of all possible outcomes of flipping two coins is

$$\Omega = \big\{(h,h),(h,t),(t,h),(t,t)\big\}.$$

Notation for Events Represented by the Random Variable X ("number of heads")

$\{X \in \Omega'\}$	$= X^{-1}(\Omega')$	$= \Omega$	0, 1, or 2 *heads*
$\{X \in \emptyset\}$	$= X^{-1}(\emptyset)$	$= \emptyset$	neither 0, 1, nor 2 *heads*
$\{X = 0\}$	$= X^{-1}(\{0\})$	$= \{(t,t)\}$	no *heads*
$\{X = 1\}$	$= X^{-1}(\{1\})$	$= \{(h,t),(t,h)\}$	one and only one *heads*
$\{X = 2\}$	$= X^{-1}(\{2\})$	$= \{(h,h)\}$	two heads
$\{X \in \{0,1\}\}$	$= X^{-1}(\{0,1\})$	$= \{(h,t),(t,h),(t,t)$	not more than one *heads*
$\{X \in \{0,2\}\}$	$= X^{-1}(\{0,2\})$	$= \{(h,h),(t,t)\}$	two *heads* or no *heads*
$\{X \in \{1,2\}\}$	$= X^{-1}(\{1,2\})$	$= \{(h,h),(h,t),(t,h)\}$	at least one *heads*

The set

$$\Big\{\Omega', \emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\Big\}$$

is a σ-algebra on Ω', and the set

$$\Big\{\Omega, \emptyset, X^{-1}(\{0\}), X^{-1}(\{1\}), X^{-1}(\{2\}), X^{-1}(\{0,1\}), X^{-1}(\{0,2\}), X^{-1}(\{1,2\})\Big\}$$

is a σ-algebra on Ω.

Definition of a random variable

Remember, the pair $(\Omega', \mathscr{A}')$ of a nonempty set Ω' and a σ-algebra $\mathscr{A}'$ on Ω' is called a *measurable space.*

Definition 1.
Let $(\Omega, \mathscr{A}, P)$ be a probability space and $(\Omega', \mathscr{A}')$ a measurable space. If $X : \Omega \to \Omega'$ satisfies

$$X^{-1}(A') \in \mathscr{A}, \quad \forall A' \in \mathscr{A}', \tag{1}$$

then X is called a *random variable on* $(\Omega, \mathscr{A}, P)$ *with values in* $(\Omega', \mathscr{A}')$.

The set

$$X^{-1}(\mathscr{A}') := \{X^{-1}(A') : A' \in \mathscr{A}'\}, \tag{2}$$

is called the σ-*algebra generated by* X. This set is also denoted by $\sigma(X)$.

If X is a random variable on $(\Omega, \mathscr{A}, P)$ with values in $(\Omega', \mathscr{A}')$, then $\sigma(X) \subset \mathscr{A}$. Hence, each element of $\sigma(X)$ is an event. That is, each element of $\sigma(X)$ is also an element of $\mathscr{A}$ and has a probability that is determined by the probability measure P on $\mathscr{A}$. (Later, this is used in the definition of the *distribution* of a random variable.)

Joe and Ann With Random Variables

Elements of Ω			Random variables			
Unit	Treatment	Success	Observational-unit variable U	Treatment variable X	Outcome variable Y	Probabilities of elementary events $P(\{\omega\})$
(*Joe*,	*no*,	$-$)	*Joe*	0	0	.09
(*Joe*,	*no*,	$+$)	*Joe*	0	1	.21
(*Joe*,	*yes*,	$-$)	*Joe*	1	0	.04
(*Joe*,	*yes*,	$+$)	*Joe*	1	1	.16
(*Ann*,	*no*,	$-$)	*Ann*	0	0	.24
(*Ann*,	*no*,	$+$)	*Ann*	0	1	.06
(*Ann*,	*yes*,	$-$)	*Ann*	1	0	.16
(*Ann*,	*yes*,	$+$)	*Ann*	1	1	.04

The probability space is $(\Omega, \mathscr{A}, P)$ with $\mathscr{A} = \mathscr{P}(\Omega)$, where Ω and P are specified in the table.

U, X, and Y are random variables on $(\Omega, \mathscr{A}, P)$.

U is with values in $(\Omega_U, \mathscr{A}_U)$, where $\Omega_U = \{Joe, Ann\}$ and $\mathscr{A}_U = \big\{\Omega_U, \emptyset, \{Joe\}, \{Ann\}\big\}$.

X and Y are with values in $(\Omega', \mathscr{A}')$, where $\Omega' = \{0,1\}$ and $\mathscr{A}' = \big\{\Omega', \emptyset, \{0\}, \{1\}\big\}$.

σ-Algebra Generated by a Set System

Definition 2 (σ-Algebra Generated by a Set System)**.**
Let $\mathscr{E}$ be a set system on Ω and let $(\mathscr{A}_i, i \in I)$ be the family of all σ-algebras on Ω that contain $\mathscr{E}$ as a subset. Then we define

$$\sigma(\mathscr{E}) := \bigcap_{i \in I} \mathscr{A}_i \qquad (3)$$

and call it the σ-*algebra generated by* $\mathscr{E}$. The set $\mathscr{E}$ is also called a *generating system of* $\sigma(\mathscr{E})$.

Exercise
Let Ω be a nonempty set and $A \subset \Omega$. Determine the σ-algebra generated by the set system $\{A\}$. As an example consider $\Omega = \mathbb{R}$ and $A = \{1\}$.

For more details and examples see chapters 1 and 2 of Steyer & Nagel (in press).

Borel σ-algebra , Real-Valued and Numerical Random Variables

Definition 3 (Borel σ-Algebra)**.**
The σ-algebra generated by the set of all open intervals of the set $\mathbb{R}$ of real numbers is called the *Borel* σ*-algebra on* $\mathbb{R}$. It is denoted by $\mathscr{B}$.

The Borel σ-algebra $\mathscr{B}$ contains as elements all open, half-open, and closed intervals, as well as their finite and countable unions and intersections. This includes all singletons $\{x\}$, $x \in \mathbb{R}$.

If X is a random variable on $(\Omega, \mathscr{A}, P)$ with values in $(\mathbb{R}, \mathscr{B})$, then it is called *real-valued.*

If X is with values in $(\bar{\mathbb{R}}, \bar{\mathscr{B}})$, then it is called *numerical,* where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ denotes the *extended set of real numbers* and $\bar{\mathscr{B}}$ is the *Borel* σ*-algebra on* $\bar{\mathbb{R}}$, i.e.,
$\bar{\mathscr{B}} := \sigma(\mathscr{B} \cup \{-\infty\} \cup \{\infty\})$.

Joe and Ann With Random Variables – continued

In the Joe-Ann-example, X and Y are also random variables on $(\Omega, \mathscr{A}, P)$ with values in $(\mathbb{R}, \mathscr{B})$, where $\mathscr{B}$ denotes the *Borel* σ*-algebra on* $\mathbb{R}$.

Exercise

Consider the closed interval $[1, 10]$ and determine its inverse image $X^{-1}([1, 10])$. Repeat this for two self-chosen intervals, their union, and their intersection.

Independence of two random variables

Definition 4. Let X and Y be a random variable on $(\Omega, \mathscr{A}, P)$ with values in $(\Omega'_X, \mathscr{A}'_X)$ and $(\Omega'_Y, \mathscr{A}'_Y)$, respectively. Then X and Y are called *P-independent* or *(stochastically) independent (with respect to P)*, if

$$P(A \cap B) = P(A) \cdot P(B) \qquad \text{for all } (A, B) \in X^{-1}(\mathscr{A}'_X) \times Y^{-1}(\mathscr{A}'_Y).$$

Independence of Random Variables

Remember, in the last lecture we defined independence of the sets $\mathscr{E}'_1, \ldots, \mathscr{E}'_n$ of events.

Definition 5. Let $(\Omega, \mathscr{A}, P)$ be a probability space. The sets of events $\mathscr{E}_1, \ldots, \mathscr{E}_n \subset \mathscr{A}$ are called *P-independent* or *(stochastically) independent with respect to P*, if all n-tupels $(A_1, \ldots, A_n) \in \mathscr{E}_1 \times \ldots \times \mathscr{E}_n$ are P-independent.

Note that this definition also applies to σ-algebras $\mathscr{A}_1, \ldots, \mathscr{A}_n \subset \mathscr{A}$. This concept is used in the following definition.

Definition 6. Let $X_1 : \Omega \to \Omega'_1, \ldots, X_n : \Omega \to \Omega'_n$ be random variables on $(\Omega, \mathscr{A}, P)$ with values in $(\Omega'_1, \mathscr{A}'_1), \ldots, (\Omega'_n, \mathscr{A}'_n)$, respectively. Then $X_1, \ldots, X_n$ are called *P-independent*, if the σ-algebras $X_1^{-1}(\mathscr{A}'_1), \ldots, X_n^{-1}(\mathscr{A}'_n)$ are P-independent.

Joe and Ann With Random Variables

Elements ω of Ω			Random variables			
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(*Ann*,	*no*,	−)	*Ann*	0	0	.24
(*Ann*,	*no*,	+)	*Ann*	0	1	.06
(*Ann*,	*yes*,	−)	*Ann*	1	0	.16
(*Ann*,	*yes*,	+)	*Ann*	1	1	.04

Exercise

Show that X and U are independent in this example. For simplicity, assume that U is with values in $(\Omega_U, \mathscr{A}_U)$, where $\Omega_U = \{Joe, Ann\}$ and $\mathscr{A}_U = \big\{\Omega_U, \emptyset, \{Joe\}, \{Ann\}\big\}$, and that X is with values in $(\Omega', \mathscr{A}')$, where $\Omega' = \{0, 1\}$ and $\mathscr{A}' = \big\{\Omega', \emptyset, \{0\}, \{1\}\big\}$.

How can we secure in a real-life example of such an experiment that X and U are independent?

Joe and Ann With Self-Selection and Random Variables

Table 1: Joe and Ann With Self-Selection

Outcomes ω				Observables		
Unit	Treatment	Success	$P(\{\omega\})$	Person variable U	Treatment variable X	Outcome variable Y
(*Joe*,	*no*,	−)	.144	*Joe*	0	0
(*Joe*,	*no*,	+)	.336	*Joe*	0	1
(*Joe*,	*yes*,	−)	.004	*Joe*	1	0
(*Joe*,	*yes*,	+)	.016	*Joe*	1	1
(*Ann*,	*no*,	−)	.096	*Ann*	0	0
(*Ann*,	*no*,	+)	.024	*Ann*	0	1
(*Ann*,	*yes*,	−)	.228	*Ann*	1	0
(*Ann*,	*yes*,	+)	.152	*Ann*	1	1

Exercise

Show that X and U are not independent in this example with self-selection to treatment conditions. For simplicity, assume that U is with values in $(\Omega_U, \mathscr{A}_U)$, where $\Omega_U = \{Joe, Ann\}$ and $\mathscr{A}_U = \big\{\Omega_U, \emptyset, \{Joe\}, \{Ann\}\big\}$, and that X is with values in $(\Omega', \mathscr{A}')$, where $\Omega' = \{0, 1\}$ and $\mathscr{A}' = \big\{\Omega', \emptyset, \{0\}, \{1\}\big\}$.