Methods of Evaluation Research
Lectures 5 and 6: Conditional Intercept and Conditional Effect Functions

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Motivation

Oftentimes the effects of a treatment differ for different individuals. Hence, we are interested in the optimal treatment for an individual or at least for individuals with a given pattern of covariates, i.e., pre-treatment variables that represent properties of the individuals prior to treatment. This question is well-known and studied, e.g., under the label "moderator analysis". However, almost all approaches treat this problem in the framework of the General Linear Model, assuming that treatment and covariates (moderator variables) are fixed (not stochastic). In our research program we also consider stochastic treatment variables and covariates as well as latent covariates and outcome variables. This distinction between stochastic and fixed covariates has implications for statistical analysis, in particular for the analysis of average treatment effects, i.e., the expectation of the conditional treatment effects over the distribution of the covariates. It has also effects for re-aggregations (other than the unconditional expectation) of the original conditional treatment effects. Similarly, the distinction between manifest and latent covariates has implications for the analysis of conditional and average treatment effects.

Nonorthogonal ANOVA

Table 1: Expectations in three treatment conditions

treatment	expectation of Y in the treatment conditions $E(Y\,|\,X{=}x)$	treatment probabilities $P(X{=}x)$
$X{=}0$ (control)	111.25	1/3
$X{=}1$ (treatment 1)	100.00	1/3
$X{=}2$ (treatment 2)	114.25	1/3
$E(Y)$	108.50	

Nonorthogonal ANOVA

Table 2: Expectations $E(Y \mid X{=}x, Z{=}z)$ in treatment × neediness conditions

treat-ment	neediness						$P(X{=}x)$
	low ($Z{=}0$)		medium ($Z{=}1$)		high ($Z{=}2$)		
$X{=}0$	120	(20/120)	110	(17/120)	60	(3/120)	(40/120)
$X{=}1$	100	(7/120)	100	(26/120)	100	(7/120)	(40/120)
$X{=}2$	80	(3/120)	90	(17/120)	140	(20/120)	(40/120)
$P(Z{=}z)$		(30/120)		(60/120)		(30/120)	

Note. Probabilities $P(X{=}x, Z{=}z)$, $P(Z{=}z)$, and $P(X{=}x)$ in parentheses.

Generalized ANCOVA

Generalized ANCOVA Model

If the treatment variable X takes values 0 or 1 indicating with its values 1 and 0 whether or not the person is treated, then the fundamental equation for generalized analysis of covariance is:

$$E(Y \mid X, Z) \;=\; g_0(Z) + g_1(Z) \cdot X, \tag{1}$$

where the *intercept function* $g_0(Z)$ and the *effect function* $g_1(Z)$ are unknown functions of the (possibly multivariate, numerical or non-numerical) covariate Z.

If Z univariate, then *traditional analysis of covariance* assumes

$$E(Y \mid X, Z) \;=\; \gamma_{00} + \gamma_{01} \cdot Z + \gamma_{10} \cdot X. \tag{2}$$

Generalized ANCOVA Model

Assuming that the treatment variable X takes values 0 or 1, the fundamental equation for generalized analysis of covariance is:

$$E(Y \mid X, Z) \;=\; g_0(Z) + g_1(Z) \cdot X, \tag{3}$$

where the *intercept function* $g_0(Z)$ and the *effect functions* $g_1(Z)$ are unknown functions of the (possibly multivariate, numerical or non-numerical) covariate Z.

If X is dichotomous, then this equation is *always* true as long as no restrictive assumptions about the intercept and/or effect functions are introduced.

Conditioning on the covariate, Equation (**??**) yields

$$E^{Z=z}(Y \mid X) \;=\; g_0(z) + g_1(z) \cdot X. \tag{4}$$

This equation shows that the effects of the treatments may be different for different values of the covariate.

Generalized ANCOVA Model

Conditioning on the treatment, Equation (**??**) yields, for $X=0$

$$E^{X=0}(Y \mid Z) \;=\; g_0(Z), \tag{5}$$

and for $X=1$

$$E^{X=1}(Y \mid Z) \;=\; g_0(Z) + g_1(Z). \tag{6}$$

Hence,

$$g_1(Z) \;=\; E^{X=1}(Y \mid Z) - E^{X=0}(Y \mid Z) \tag{7}$$

is the effect function, comparing treatment 1 to treatment 0 and

$$E\left[g_1(Z)\right] \;=\; E\left[E^{X=1}(Y \mid Z)\right] - E\left[E^{X=0}(Y \mid Z)\right] \tag{8}$$

is the *average effect* of treatment 1 compared to treatment 0.

If $x = 0, 1$ denote the values of X, in generalized ANCOVA, we estimate both the conditional effect functions $g_1(Z)$ and the average effects $E[g_1(Z)]$.

Linear Effect Function

Suppose that $g_1(Z)$ is a linear function of Z, then

$$E\left[g_1(Z)\right] = E\left(\gamma_{10} + \gamma_{11} Z\right) = \gamma_{10} + \gamma_{11} E(Z). \quad (9)$$

Re-aggregation

$E\left[g_1(Z)\right]$ is an example of re-aggregation of Z-conditional effects. Suppose that $Z = (Z_1, \ldots, Z_m)$ is an m-variate covariate, then

$$E\left[g_1(Z) \,\middle|\, Z_1\right]$$

is another example of a re-aggregation of Z-conditional effects.

$J+1$ **Treatment Conditions** 11 / **??**

Generalized ANCOVA Model

If the treatment variable X takes values $0, 1, \ldots, J$, and the random variables $1_{X=x}$ indicate with their values 1 and 0 whether or not $X{=}x$, then the fundamental equation for generalized analysis of covariance is:

$$E(Y \mid X, Z) \;=\; g_0(Z) + \sum_{x=1}^{J} g_x(Z) \cdot 1_{X=x}, \tag{10}$$

where the *intercept function* $g_0(Z)$ and the *effect functions* $g_x(Z)$ are unknown functions of the (possibly multivariate, numerical or non-numerical) covariate Z.

Remember, if Z univariate, then *traditional analysis of covariance* assumes

$$E(Y \mid X, Z) \;=\; \gamma_{00} + \gamma_{01} \cdot Z + \sum_{x=1}^{J} \gamma_{x0} \cdot 1_{X=x}. \tag{11}$$

Generalized ANCOVA Model

Assuming that the treatment variable X takes values $0, 1, \ldots, J$, the fundamental equation for generalized analysis of covariance is:

$$E(Y \mid X, Z) \;=\; g_0(Z) + \sum_{x=1}^{J} g_x(Z) \cdot 1_{X=x}, \tag{12}$$

where the *intercept function* $g_0(Z)$ and the *effect functions* $g_x(Z)$ are unknown functions of the (possibly multivariate, numerical or non-numerical) covariate Z.

If X is discrete this equation is *always* true as long as no restrictive assumptions about the intercept and/or effect functions are introduced.

Conditioning on the covariate, Equation (**??**) yields

$$E^{Z=z}(Y \mid X) \;=\; g_0(z) + \sum_{x=1}^{J} g_x(z) \cdot 1_{X=x}. \tag{13}$$

This equation shows that the effects of the treatments may be different for different values of the covariate.

Generalized ANCOVA Model

Conditioning on the treatment, Equation (**??**) yields, for $X=0$

$$E^{X=0}(Y \mid Z) \;=\; g_0(Z), \tag{14}$$

and for $X=x$:

$$E^{X=x}(Y \mid Z) \;=\; g_0(Z) + g_x(Z). \tag{15}$$

Hence,

$$g_x(Z) \tag{16}$$

is the effect function, comparing treatment x to treatment 0 and

$$E[g_x(Z)] \tag{17}$$

is the *average effect* of treatment x compared to treatment 0.

If $x = 0, \ldots, J$ denote the values of X, in generalized ANCOVA, we estimate both the conditional-effect functions $g_x(Z)$ and the average effects $E[g_x(Z)]$, for $x = 1, \ldots, J$.